Solutions to

Unit 3B Specialist Mathematics

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Preface

The answers in the Sadler text book sometimes are not enough. For those times when your really need to see a fully worked solution, look here.

It is essential that you use this sparingly!

You should not look here until you have given your best effort to a problem. Understand the problem here, then go away and do it on your own.

Contents

Chapter 1

Exercise 1A

1. $$\begin{aligned}\sqrt{-25} &= \sqrt{25\times -1}\\ &= \sqrt{25}\times\sqrt{-1}\\ &= 5\mathrm{i}\end{aligned}$$

2. $$\begin{aligned}\sqrt{-144} &= \sqrt{144\times -1}\\ &= \sqrt{144}\times\sqrt{-1}\\ &= 12\mathrm{i}\end{aligned}$$

3. $$\begin{aligned}\sqrt{-9} &= \sqrt{9\times -1}\\ &= \sqrt{9}\times\sqrt{-1}\\ &= 3\mathrm{i}\end{aligned}$$

4. $$\begin{aligned}\sqrt{-49} &= \sqrt{49\times -1}\\ &= \sqrt{49}\times\sqrt{-1}\\ &= 7\mathrm{i}\end{aligned}$$

5. $$\begin{aligned}\sqrt{-400} &= \sqrt{400\times -1}\\ &= \sqrt{400}\times\sqrt{-1}\\ &= 20\mathrm{i}\end{aligned}$$

6. $$\begin{aligned}\sqrt{-5} &= \sqrt{5\times -1}\\ &= \sqrt{5}\times\sqrt{-1}\\ &= \sqrt{5}\mathrm{i}\end{aligned}$$

7. $$\begin{aligned}\sqrt{-8} &= \sqrt{8\times -1}\\ &= \sqrt{8}\times\sqrt{-1}\\ &= 2\sqrt{2}\mathrm{i}\end{aligned}$$

8. $$\begin{aligned}\sqrt{-45} &= \sqrt{45\times -1}\\ &= \sqrt{9\times 5}\times\sqrt{-1}\\ &= 3\sqrt{5}\mathrm{i}\end{aligned}$$

No working required for questions 9, 10 and 11.

12. $$\begin{aligned}x &= \frac{-2\pm\sqrt{2^2-4\times1\times5}}{2\times1}\\ &= \frac{-2\pm\sqrt{4-20}}{2}\\ &= \frac{-2\pm\sqrt{-16}}{2}\\ &= \frac{-2\pm4\mathrm{i}}{2}\\ x &= -1+2\mathrm{i} \quad\text{or}\quad x=-1-2\mathrm{i}\end{aligned}$$

13. $$\begin{aligned}x &= \frac{-2\pm\sqrt{2^2-4\times1\times3}}{2\times1}\\ &= \frac{-2\pm\sqrt{4-12}}{2}\\ &= \frac{-2\pm\sqrt{-8}}{2}\\ &= \frac{-2\pm2\sqrt{2}\mathrm{i}}{2}\\ x &= -1+\sqrt{2}\mathrm{i} \quad\text{or}\quad x=-1-\sqrt{2}\mathrm{i}\end{aligned}$$

14. $$\begin{aligned}x &= \frac{-4\pm\sqrt{4^2-4\times1\times6}}{2\times1}\\ &= \frac{-4\pm\sqrt{16-24}}{2}\\ &= \frac{-4\pm\sqrt{-8}}{2}\\ &= \frac{-4\pm2\sqrt{2}\mathrm{i}}{2}\\ x &= -2+\sqrt{2}\mathrm{i} \quad\text{or}\quad x=-2-\sqrt{2}\mathrm{i}\end{aligned}$$

15. $$\begin{aligned}x &= \frac{-2\pm\sqrt{2^2-4\times1\times10}}{2\times1}\\ &= \frac{-2\pm\sqrt{4-40}}{2}\\ &= \frac{-2\pm\sqrt{-36}}{2}\\ &= \frac{-2\pm6\mathrm{i}}{2}\\ x &= -1+3\mathrm{i} \quad\text{or}\quad x=-1-3\mathrm{i}\end{aligned}$$

16. $$\begin{aligned}x &= \frac{4\pm\sqrt{4^2-4\times1\times6}}{2\times1}\\ &= \frac{4\pm\sqrt{16-24}}{2}\\ &= \frac{4\pm\sqrt{-8}}{2}\\ &= \frac{4\pm2\sqrt{2}\mathrm{i}}{2}\\ x &= 2+\sqrt{2}\mathrm{i} \quad\text{or}\quad x=2-\sqrt{2}\mathrm{i}\end{aligned}$$

17. $$\begin{aligned}x &= \frac{1\pm\sqrt{(-1)^2-4\times2\times1}}{2\times2}\\ &= \frac{1\pm\sqrt{1-8}}{4}\\ &= \frac{1\pm\sqrt{-7}}{4}\\ &= \frac{1\pm\sqrt{7}\mathrm{i}}{4}\\ x &= \frac{1}{4}+\frac{\sqrt{7}}{4}\mathrm{i} \quad\text{or}\quad x=\frac{1}{4}-\frac{\sqrt{7}}{4}\mathrm{i}\end{aligned}$$

18. $$\begin{aligned}x &= \frac{-1\pm\sqrt{1^2-4\times2\times1}}{2\times2}\\ &= \frac{-1\pm\sqrt{1-8}}{4}\\ &= \frac{-1\pm\sqrt{-7}}{4}\\ &= \frac{-1\pm\sqrt{7}\mathrm{i}}{4}\\ x &= -\frac{1}{4}+\frac{\sqrt{7}}{4}\mathrm{i} \quad\text{or}\quad x=-\frac{1}{4}-\frac{\sqrt{7}}{4}\mathrm{i}\end{aligned}$$

19. $$\begin{aligned} x &= \frac{-6 \pm \sqrt{6^2 - 4 \times 2 \times 5}}{2 \times 2} \\ &= \frac{-6 \pm \sqrt{36 - 40}}{4} \\ &= \frac{-6 \pm \sqrt{-4}}{4} \\ &= \frac{-6 \pm 2\mathrm{i}}{4} \\ x &= -\frac{3}{2} + \frac{1}{2}\mathrm{i} \quad \text{or} \quad x = -\frac{3}{2} - \frac{1}{2}\mathrm{i} \end{aligned}$$

20. $$\begin{aligned} x &= \frac{2 \pm \sqrt{(-2)^2 - 4 \times 2 \times 25}}{2 \times 2} \\ &= \frac{2 \pm \sqrt{4 - 200}}{4} \\ &= \frac{2 \pm \sqrt{-196}}{4} \\ &= \frac{2 \pm 14\mathrm{i}}{4} \\ x &= \frac{1}{2} + \frac{7}{2}\mathrm{i} \quad \text{or} \quad x = \frac{1}{2} - \frac{7}{2}\mathrm{i} \end{aligned}$$

21. $$\begin{aligned} x &= \frac{2 \pm \sqrt{(-2)^2 - 4 \times 5 \times 13}}{2 \times 5} \\ &= \frac{2 \pm \sqrt{4 - 260}}{10} \\ &= \frac{2 \pm \sqrt{-256}}{10} \\ &= \frac{2 \pm 16\mathrm{i}}{10} \\ x &= 0.2 + 1.6\mathrm{i} \quad \text{or} \quad x = 0.2 - 1.6\mathrm{i} \end{aligned}$$

22. $$\begin{aligned} x &= \frac{1 \pm \sqrt{(-1)^2 - 4 \times 1 \times 1}}{2 \times 1} \\ &= \frac{1 \pm \sqrt{1 - 4}}{2} \\ &= \frac{1 \pm \sqrt{-3}}{2} \\ &= \frac{1 \pm \sqrt{3}\mathrm{i}}{2} \\ x &= \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i} \quad \text{or} \quad x = \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i} \end{aligned}$$

23. $$\begin{aligned} x &= \frac{3 \pm \sqrt{(-3)^2 - 4 \times 5 \times 1}}{2 \times 5} \\ &= \frac{3 \pm \sqrt{9 - 20}}{10} \\ &= \frac{3 \pm \sqrt{-11}}{10} \\ &= \frac{3 \pm \sqrt{11}\mathrm{i}}{10} \\ x &= 0.3 + \frac{\sqrt{11}}{10}\mathrm{i} \quad \text{or} \quad x = 0.3 - \frac{\sqrt{11}}{10}\mathrm{i} \end{aligned}$$

Exercise 1B

1. $(2 + 5) + (3 - 1)\mathrm{i} = 7 + 2\mathrm{i}$
2. $(5 - 2) + (-6 - 4)\mathrm{i} = 3 - 10\mathrm{i}$
3. $(2 - 5) + (3 - -1)\mathrm{i} = -3 + 4\mathrm{i}$
4. $(5 + 2) + (-6 + 4)\mathrm{i} = 7 - 2\mathrm{i}$
5. $(2 - 5) + (3 - 1)\mathrm{i} = -3 + 2\mathrm{i}$
6. $(5 + 2) + (-6 + 4)\mathrm{i} = 7 - 2\mathrm{i}$
7. $(3 + 4 + 6) + (1 - 2 + 5)\mathrm{i} = 13 + 4\mathrm{i}$
8. $(6 + 4\mathrm{i}) + (6 + 3\mathrm{i}) = (6 + 6) + (4 + 3)\mathrm{i} = 12 + 7\mathrm{i}$
9. $(10 + 5\mathrm{i}) + (3 - 3\mathrm{i}) = (10 + 3) + (5 - 3)\mathrm{i} = 13 + 2\mathrm{i}$
10. $(10 + 5\mathrm{i}) - (3 - 3\mathrm{i}) = (10 - 3) + (5 - -3)\mathrm{i} = 7 + 8\mathrm{i}$
11. $(3 - 15\mathrm{i}) + 7\mathrm{i} = 3 + (-15 + 7)\mathrm{i} = 3 - 8\mathrm{i}$
12. $(3 - 15\mathrm{i}) + 7 = (3 + 7) - 15\mathrm{i} = 10 - 15\mathrm{i}$
13. $2 + 5 = 7$
14. $4 + 1 = 5$
15. $6 + 15\mathrm{i} + 4\mathrm{i} + 10\mathrm{i}^2 = 6 + 19\mathrm{i} - 10 = -4 + 19\mathrm{i}$
16. $3 + 2\mathrm{i} + 9\mathrm{i} + 6\mathrm{i}^2 = 3 + 11\mathrm{i} - 6 = -3 + 11\mathrm{i}$
17. $2 - 2\mathrm{i} + \mathrm{i} - \mathrm{i}^2 = 2 - \mathrm{i} - -1 = 3 - \mathrm{i}$
18. $-10 - 2\mathrm{i} + 15\mathrm{i} + 3\mathrm{i}^2 = -10 + 13\mathrm{i} - 3 = -13 + 13\mathrm{i}$

19. $$\begin{aligned}\frac{3+2\mathrm{i}}{1+5\mathrm{i}} &= \frac{3+2\mathrm{i}}{1+5\mathrm{i}} \times \frac{1-5\mathrm{i}}{1-5\mathrm{i}}\\ &= \frac{3-15\mathrm{i}+2\mathrm{i}-10\mathrm{i}^2}{1^2-(5\mathrm{i})^2}\\ &= \frac{3-13\mathrm{i}--10}{1--25}\\ &= \frac{13-13\mathrm{i}}{26}\\ &= 0.5+0.5\mathrm{i}\end{aligned}$$

20. $$\begin{aligned}\frac{3+\mathrm{i}}{1-2\mathrm{i}} &= \frac{3+\mathrm{i}}{1-2\mathrm{i}} \times \frac{1+2\mathrm{i}}{1+2\mathrm{i}}\\ &= \frac{3+6\mathrm{i}+\mathrm{i}+2\mathrm{i}^2}{1^2-(2\mathrm{i})^2}\\ &= \frac{3+7\mathrm{i}+-2}{1--4}\\ &= \frac{1+7\mathrm{i}}{5}\\ &= 0.2+1.4\mathrm{i}\end{aligned}$$

21. $$\begin{aligned}\frac{4}{1+3\mathrm{i}} &= \frac{4}{1+3\mathrm{i}} \times \frac{1-3\mathrm{i}}{1-3\mathrm{i}}\\ &= \frac{4-12\mathrm{i}}{1^2-(3\mathrm{i})^2}\\ &= \frac{4-12\mathrm{i}}{1--9}\\ &= \frac{4-12\mathrm{i}}{10}\\ &= 0.4+1.2\mathrm{i}\end{aligned}$$

22. $$\begin{aligned}\frac{2\mathrm{i}}{1+4\mathrm{i}} &= \frac{2\mathrm{i}}{1+4\mathrm{i}} \times \frac{1-4\mathrm{i}}{1-4\mathrm{i}}\\ &= \frac{2\mathrm{i}-8\mathrm{i}^2}{1^2-(4\mathrm{i})^2}\\ &= \frac{2\mathrm{i}--8}{1--16}\\ &= \frac{8+2\mathrm{i}}{17}\\ &= \frac{8}{17}+\frac{2}{17}\mathrm{i}\end{aligned}$$

23. $$\begin{aligned}\frac{-3+2\mathrm{i}}{2+3\mathrm{i}} &= \frac{-3+2\mathrm{i}}{2+3\mathrm{i}} \times \frac{2-3\mathrm{i}}{2-3\mathrm{i}}\\ &= \frac{-6+9\mathrm{i}+4\mathrm{i}-6\mathrm{i}^2}{2^2-(3\mathrm{i})^2}\\ &= \frac{-6+13\mathrm{i}--6}{4--9}\\ &= \frac{13\mathrm{i}}{13}\\ &= \mathrm{i}\end{aligned}$$

24. $$\begin{aligned}\frac{5+\mathrm{i}}{2\mathrm{i}+3} &= \frac{5+\mathrm{i}}{2\mathrm{i}+3} \times \frac{-2\mathrm{i}+3}{-2\mathrm{i}+3}\\ &= \frac{-10\mathrm{i}+15-2\mathrm{i}^2+3\mathrm{i}}{-(2\mathrm{i})^2+3^2}\\ &= \frac{15-7\mathrm{i}--2}{--4+9}\\ &= \frac{17-7\mathrm{i}}{13}\\ &= \frac{17}{13}-\frac{7}{13}\mathrm{i}\end{aligned}$$

25. (a) $w+z=(5+4)+(-2+3)\mathrm{i}=9+\mathrm{i}$

(b) $w-z-(5-4)+(-2-3)\mathrm{i}=1-5\mathrm{i}$

(c) $3w-2z=(15-6\mathrm{i})-(8+6\mathrm{i})=(15-8)+(-6-6)\mathrm{i}=7-12\mathrm{i}$

(d) $wz=(5-2\mathrm{i})(4+3\mathrm{i})=20+15\mathrm{i}-8\mathrm{i}-6\mathrm{i}^2=26+7\mathrm{i}$

(e) $z^2=(4+3\mathrm{i})^2=16+24\mathrm{i}+9\mathrm{i}^2=7+24\mathrm{i}$

(f) $$\begin{aligned}\frac{w}{z} &= \frac{5-2\mathrm{i}}{4+3\mathrm{i}} \times \frac{4-3\mathrm{i}}{4-3\mathrm{i}}\\ &= \frac{20-15\mathrm{i}-8\mathrm{i}+6\mathrm{i}^2}{4^2-(3\mathrm{i})^2}\\ &= \frac{20-23\mathrm{i}+-6}{16--9}\\ &= \frac{14-23\mathrm{i}}{25}\\ &= 0.56-0.92\mathrm{i}\end{aligned}$$

26. (a) $Z_1+Z_2=(3+1)+(5-5)\mathrm{i}=4$

(b) $Z_2-Z_1=(1-3)+(-5-5)\mathrm{i}=-2-10\mathrm{i}$

(c) $Z_1+3Z_2=(3+5\mathrm{i})+(3-15\mathrm{i})=(3+3)+(5-15)\mathrm{i}=6-10\mathrm{i}$

(d) $Z_1Z_2=(3+5\mathrm{i})(1-5\mathrm{i})=3-15\mathrm{i}+5\mathrm{i}-25\mathrm{i}^2=28-10\mathrm{i}$

(e) $Z_1^2=(3+5\mathrm{i})^2=9+30\mathrm{i}+25\mathrm{i}^2=-16+30\mathrm{i}$

(f) $$\begin{aligned}\frac{Z_1}{Z_2} &= \frac{3+5\mathrm{i}}{1-5\mathrm{i}} \times \frac{1+5\mathrm{i}}{1+5\mathrm{i}}\\ &= \frac{3+15\mathrm{i}+5\mathrm{i}+25\mathrm{i}^2}{1^2-(5\mathrm{i})^2}\\ &= \frac{3+20\mathrm{i}+-25}{1--25}\\ &= \frac{-22+20\mathrm{i}}{26}\\ &= -\frac{11}{13}+\frac{10}{13}\mathrm{i}\end{aligned}$$

27. (a) $\bar{z}=24+7\mathrm{i}$

(b) $z+\bar{z}=24-7\mathrm{i}+24+7\mathrm{i}=48$

(c) $z\bar{z}=(24-7\mathrm{i})(24+7\mathrm{i})=24^2-(7\mathrm{i})^2=576--49=625$

(d) $$\begin{aligned}\frac{z}{\bar{z}} &= \frac{z}{\bar{z}} \times \frac{z}{z}\\ &= \frac{(24-7\mathrm{i})^2}{625}\\ &= \frac{576-336\mathrm{i}+49\mathrm{i}^2}{625}\\ &= \frac{527-336\mathrm{i}}{625}\\ &= \frac{527}{625}-\frac{336}{625}\mathrm{i}\end{aligned}$$

28. (a) $\bar{z} = 4 - 9\mathrm{i}$

(b) $z - \bar{z} = (4+9\mathrm{i}) - (4-9\mathrm{i}) = 18\mathrm{i}$

(c) $2z + 3\bar{z} = (8+18\mathrm{i}) + (12-27\mathrm{i}) = 20 - 9\mathrm{i}$

(d) $2z - 3\bar{z} = (8+18\mathrm{i}) - (12-27\mathrm{i}) = -4 + 45\mathrm{i}$

(e) $z\bar{z} = (4+9\mathrm{i})(4-9\mathrm{i}) = 4^2 - (9\mathrm{i})^2 = 16 - -81 = 97$

(f) $$\begin{aligned}\frac{z}{\bar{z}} &= \frac{z}{\bar{z}} \times \frac{z}{z}\\ &= \frac{(4+9\mathrm{i})^2}{97}\\ &= \frac{16+72\mathrm{i}+81\mathrm{i}^2}{97}\\ &= \frac{-65+72\mathrm{i}}{97}\\ &= -\frac{65}{97}+\frac{72}{97}\mathrm{i}\end{aligned}$$

29. $$\begin{aligned}z &= w\\ 2+c\mathrm{i} &= d+3\mathrm{i}\\ \mathrm{Re}(z) &= \mathrm{Re}(w)\\ 2 &= d\\ \mathrm{Im}(z) &= \mathrm{Im}(w)\\ c &= 3\end{aligned}$$

30. $$\begin{aligned}a+b\mathrm{i} &= (2-3\mathrm{i})^2\\ &= 4-12\mathrm{i}+9\mathrm{i}^2\\ &= 4-12\mathrm{i}-9\\ &= -5-12\mathrm{i}\\ a &= -5\\ b &= -12\end{aligned}$$

31. $$\begin{aligned}z &= w\\ 5-(c+3)\mathrm{i} &= d+1+7\mathrm{i}\\ \mathrm{Re}(z) &= \mathrm{Re}(w)\\ 5 &= d+1\\ d &= 4\\ \mathrm{Im}(z) &= \mathrm{Im}(w)\\ -(c+3) &= 7\\ c+3 &= -7\\ c &= -10\end{aligned}$$

32. $$\begin{aligned}(a+3\mathrm{i})(5-\mathrm{i}) &= p\\ 5a-a\mathrm{i}+15\mathrm{i}-3\mathrm{i}^2 &= p\\ 5a+3+(15-a)\mathrm{i} &= p\\ 15-a &= 0\\ a &= 15\\ 5a+3 &= p\\ 75+3 &= p\\ p &= 78\end{aligned}$$

33. (a) Yes, this is true. This is how the conjugate is defined: real parts equal, imaginary parts opposite.

(b) No, this is not necessarily true. For example, consider $z = 1 + 3\mathrm{i}$ and $w = 2 - 3\mathrm{i}$. Here $\mathrm{Im}(z) = -\mathrm{Im}(w)$ but $\mathrm{Re}(z) \neq \mathrm{Re}(w)$ so $w \neq \bar{z}$

34. (a) In the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

the value of the square root $\sqrt{b^2-4ac}$ either zero (for $b^2-4ac = 0$), real ($b^2-4ac > 0$) or imaginary ($b^2-4ac < 0$). If zero or real, the quadratic has one or two real roots. If it has complex roots we must have $\sqrt{b^2-4ac} = q\mathrm{i}$ for some real q. The two roots, then are

$$x = \frac{-b+q\mathrm{i}}{2a} = \frac{-b}{2a}+\frac{q}{2a}\mathrm{i} \quad \text{and} \quad x = \frac{-b-q\mathrm{i}}{2a} = \frac{-b}{2a}-\frac{q}{2a}\mathrm{i}$$

from which we can see that the real parts are equal, and the imaginary parts are opposites, that is they are conjugates. □

(b) From the quadratic formula,

$$3+2\mathrm{i} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Substituting $a = 1$ and equating real parts we get

$$\begin{aligned}3 &= \frac{-b}{2a}\\ &= \frac{-b}{2}\\ b &= -6\end{aligned}$$

Equating the imaginary parts,

$$\begin{aligned}2\mathrm{i} &= \frac{\sqrt{b^2-4ac}}{2a}\\ 2\mathrm{i} &= \frac{\sqrt{(-6)^2-4c}}{2}\\ 4\mathrm{i} &= \sqrt{36-4c}\\ (4\mathrm{i})^2 &= 36-4c\\ -16 &= 36-4c\\ -52 &= -4c\\ c &= 13\end{aligned}$$

(c) From the quadratic formula,

$$5 - 3\mathrm{i} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substituting $a = 1$ and equating real parts we get

$$\begin{aligned} 5 &= \frac{-b}{2a} \\ &= \frac{-b}{2} \\ b &= -10 \end{aligned}$$

Equating the parts,

$$\begin{aligned} -3\mathrm{i} &= \frac{\sqrt{b^2 - 4ac}}{2a} \\ -3\mathrm{i} &= \frac{\sqrt{(-10)^2 - 4c}}{2} \\ -6\mathrm{i} &= \sqrt{100 - 4c} \\ (-6\mathrm{i})^2 &= 100 - 4c \\ -36 &= 100 - 4c \\ -136 &= -4c \\ c &= 34 \end{aligned}$$

35. (a)
$$\begin{aligned} \frac{c + d\mathrm{i}}{-c - d\mathrm{i}} &= c + d\mathrm{i}{-c} - d\mathrm{i} \times -c + d\mathrm{i}{-c} + d\mathrm{i} \\ &= \frac{-c^2 + (d\mathrm{i})^2}{(-c)^2 - (d\mathrm{i})^2} \\ &= \frac{-c^2 - d^2}{c^2 + d^2} \\ &= -1 \end{aligned}$$

(b)
$$\begin{aligned} \frac{c + d\mathrm{i}}{d - c\mathrm{i}} &= c + d\mathrm{i}d - c\mathrm{i} \times d + c\mathrm{i}d + c\mathrm{i} \\ &= \frac{cd + c^2\mathrm{i} + d^2\mathrm{i} + cd\mathrm{i}^2}{(d)^2 - (c\mathrm{i})^2} \\ &= \frac{(c^2 + d^2)\mathrm{i}}{c^2 + d^2} \\ &= \mathrm{i} \end{aligned}$$

(c)
$$\begin{aligned} \frac{c - d\mathrm{i}}{-d - c\mathrm{i}} &= c - d\mathrm{i}{-d} - c\mathrm{i} \times -d + c\mathrm{i}{-d} + c\mathrm{i} \\ &= \frac{-cd + c^2\mathrm{i} + d^2\mathrm{i} - cd\mathrm{i}^2}{(-d)^2 - (c\mathrm{i})^2} \\ &= \frac{(c^2 + d^2)\mathrm{i}}{c^2 + d^2} \\ &= \mathrm{i} \end{aligned}$$

This should be expected, since we are just substituting $-d$ for d in the previous question. If the outcome is true for all d then it must also be true for all $-d$.

36.
$$\begin{aligned} \frac{3 + 5\mathrm{i}}{1 + p\mathrm{i}} &= q + 4\mathrm{i} \\ \frac{3 + 5\mathrm{i}}{1 + p\mathrm{i}} \times \frac{1 - p\mathrm{i}}{1 - p\mathrm{i}} &= q + 4\mathrm{i} \\ \frac{3 - 3p\mathrm{i} + 5\mathrm{i} - 5p\mathrm{i}^2}{1^2 - (p\mathrm{i})^2} &= q + 4\mathrm{i} \\ \frac{(3 + 5p) + (5 - 3p)\mathrm{i}}{1 + p^2} &= q + 4\mathrm{i} \\ (3 + 5p) + (5 - 3p)\mathrm{i} &= (1 + p^2)(q + 4\mathrm{i}) \end{aligned}$$

Equating imaginary components:

$$\begin{aligned} 5 - 3p &= 4(1 + p^2) \\ 4p^2 + 3p - 1 &= 0 \\ (4p - 1)(p + 1) &= 0 \\ p &= \frac{1}{4} \qquad \text{or} \qquad p = -1 \end{aligned}$$

Now real components:

$$\begin{aligned} 3 + 5p &= q(1 + p^2) \\ q &= \frac{3 + 5p}{1 + p^2} \\ q &= \frac{3 + 5(\frac{1}{4})}{1 + (\frac{1}{4})^2} \qquad \text{or} \qquad q = \frac{3 + 5(-1)}{1 + (-1)^2} \\ &= \frac{\frac{17}{4}}{\frac{17}{16}} \qquad\qquad\qquad = \frac{-2}{2} \\ &= \frac{17}{4} \times \frac{16}{17} \qquad\qquad = -1 \\ &= 4 \end{aligned}$$

Solution: $p = \frac{1}{4}$, $q = 4$ or $p = -1$, $q = -1$.

37. (a)
$$\begin{aligned} (x - z)(x - w) &= ax^2 + bx + c \\ x^2 + (-w - z)x + wz &= ax^2 + bx + c \\ a &= 1 \\ b &= (-w - z) \\ c &= wz \end{aligned}$$

(b) From the above, $b = -w - z$ so $w + z = -b$ which is real.
$wz = c$ which is real. □

(c) Since $w + z$ is real, $\mathrm{Im}(w + z) = 0$:

$$\begin{aligned} \mathrm{Im}(p + q\mathrm{i} + r + s\mathrm{i}) &= 0 \\ q + s &= 0 \\ s &= -q \end{aligned}$$

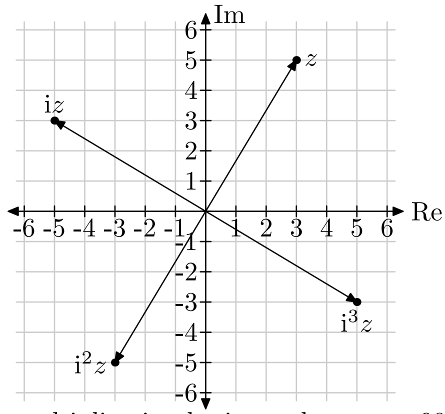
Similarly $\mathrm{Im}(wz) = 0$:

$$\begin{aligned} \mathrm{Im}\left((p + q\mathrm{i})(r + s\mathrm{i})\right) &= 0 \\ \mathrm{Im}(pr + ps\mathrm{i} + qr\mathrm{i} - qs) &= 0 \\ ps + qr &= 0 \\ p(-q) + qr &= 0 \\ -p + r &= 0 \\ r &= p \end{aligned}$$

Hence $r + s\mathrm{i} = p - q\mathrm{i}$ □

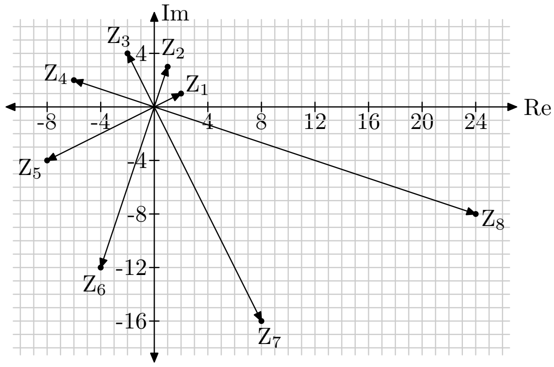
Exercise 1C

1. No working required. Refer answers in Sadler.
2. No working required. Refer answers in Sadler.
3. No working required. Refer answers in Sadler. Note that the complex conjugate produces a reflection in the Real axis on the Argand diagram (so Z_7 is a reflection of Z_1 and Z_8 of Z_3).
4.
   - $\frac{\text{Re}(Z_2)}{\text{Im}(Z_2)} > 1$ means Z_2 must be in either quadrant I or III on the Argand diagram and the real part larger in magnitude than the imaginary part. Only one number on the Argand diagram satisfies this: $Z_2 = -3 - 2\text{i}$
   - $\frac{\text{Re}(Z_1)}{\text{Im}(Z_1)} > 0$ means Z_1 must be in either quadrant I or III on the Argand diagram. Since we have already eliminated the number in quadrant III we can conclude $Z_1 = 1 + 2\text{i}$
   - $Z_3 = \bar{Z}_2 = -3 + 2\text{i}$
   - This leaves $Z_4 = 3 - 2\text{i}$
5.
   - $z = 3 + 5\text{i}$;
   - $\text{i}z = 3\text{i} + 5\text{i}^2 = -5 + 3\text{i}$;
   - $\text{i}^2 z = -z = -3 - 5\text{i}$;
   - $\text{i}^3 z = -\text{i}z = 5 - 3\text{i}$.



   Note: multiplication by i translates to a 90° rotation on the Argand diagram.

6.
   - $Z_1 = 2 + \text{i}$
   - $$\begin{aligned} Z_2 &= (2+\text{i})(1+\text{i}) \\ &= 2 + 2\text{i} + \text{i} + \text{i}^2 \\ &= 2 + 3\text{i} - 1 \\ &= 1 + 3\text{i} \end{aligned}$$
   - $$\begin{aligned} Z_3 &= (2+\text{i})(1+\text{i})^2 \\ &= Z_2(1+\text{i}) \\ &= (1+3\text{i})(1+\text{i}) \\ &= 1 + \text{i} + 3\text{i} + 3\text{i}^2 \\ &= 1 + 4\text{i} - 3 \\ &= -2 + 4\text{i} \end{aligned}$$
   - $$\begin{aligned} Z_4 &= (2+\text{i})(1+\text{i})^3 \\ &= Z_3(1+\text{i}) \\ &= (-2+4\text{i})(1+\text{i}) \\ &= -2 - 2\text{i} + 4\text{i} + 4\text{i}^2 \\ &= -2 + 2\text{i} - 4 \\ &= -6 + 2\text{i} \end{aligned}$$
   - $$\begin{aligned} Z_5 &= (2+\text{i})(1+\text{i})^4 \\ &= Z_4(1+\text{i}) \\ &= (-6+2\text{i})(1+\text{i}) \\ &= -6 - 6\text{i} + 2\text{i} + 2\text{i}^2 \\ &= -6 - 4\text{i} - 2 \\ &= -8 - 4\text{i} \end{aligned}$$
   - $$\begin{aligned} Z_6 &= (2+\text{i})(1+\text{i})^5 \\ &= Z_5(1+\text{i}) \\ &= (-8-4\text{i})(1+\text{i}) \\ &= -8 - 8\text{i} - 4\text{i} - 4\text{i}^2 \\ &= -8 - 12\text{i} + 4 \\ &= -4 - 12\text{i} \end{aligned}$$
   - $$\begin{aligned} Z_7 &= (2+\text{i})(1+\text{i})^6 \\ &= Z_6(1+\text{i}) \\ &= (-4-12\text{i})(1+\text{i}) \\ &= -4 - 4\text{i} - 12\text{i} - 12\text{i}^2 \\ &= -4 - 16\text{i} + 12 \\ &= 8 - 16\text{i} \end{aligned}$$
   - $$\begin{aligned} Z_8 &= (2+\text{i})(1+\text{i})^7 \\ &= Z_7(1+\text{i}) \\ &= (8-16\text{i})(1+\text{i}) \\ &= 8 + 8\text{i} - 16\text{i} - 16\text{i}^2 \\ &= 8 - 8\text{i} + 16 \\ &= 24 - 8\text{i} \end{aligned}$$



Miscellaneous Exercise 1

1. $\mathbf{F}+\mathbf{P} = (13\mathbf{i}-28\mathbf{j}) + (-6\mathbf{i}+4\mathbf{j})$
   $= 7\mathbf{i}-24\mathbf{j}$
   Magnitude of the resultant is $\sqrt{7^2+24^2} = 25\text{N}$

2. (a) $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \mathbf{a}+3\mathbf{b}$

   (b) $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AO} + \overrightarrow{OC}$
   $= -\overrightarrow{AB} - \overrightarrow{OA} + \overrightarrow{OC}$
   $= -3\mathbf{b} - \mathbf{a} + 7\mathbf{b}$
   $= 4\mathbf{b} - \mathbf{a}$

   (c) $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AO} + \overrightarrow{OC}$
   $= -\overrightarrow{AB} - \overrightarrow{OA} + \overrightarrow{OC}$
   $= -3\mathbf{b} - \mathbf{a} + 7\mathbf{b}$
   $= 4\mathbf{b} - \mathbf{a}$

   (d) $\overrightarrow{BD} = 0.5\overrightarrow{BC}$
   $= 0.5(4\mathbf{b}-\mathbf{a})$
   $= 2\mathbf{b} - 0.5\mathbf{a}$

   (e) $\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD}$
   $= \mathbf{a}+3\mathbf{b}+2\mathbf{b}-0.5\mathbf{a}$
   $= 0.5\mathbf{a}+5\mathbf{b}$

3. (a) $\log_a 2 = \log_a \frac{10}{5} = \log_a 10 - \log_a 5 = p - q$

   (b) $\log_a 50 = \log_a 10\times 5 = \log_a 10 + \log_a 5 = p+q$

   (c) $\log_a 100 = \log_a 10^2 = 2\log_a 10 = 2p$

   (d) $\log_a 125 = \log_a 5^3 = 3\log_a 5 = 3q$

   (e) $\log_a 0.1 = \log_a 10^{-1} = -\log_a 10 = -p$

   (f) $\log_a 0.5 = \log_a \frac{5}{10} = \log_a 5 - \log_a 10 = q - p$

   (g) $\log_a 20a = \log_a \frac{100}{5}a$
   $= \log_a 10^2 - \log_a 5 + \log_a a$
   $= 2\log_a 10 - \log_a 5 + \log_a a$
   $= 2p - q + 1$

   (h) $\log_5 10 = \frac{\log_a 10}{\log_a 5} = \frac{p}{q}$

   (i) $\log 5 = \frac{\log_a 5}{\log_a 10} = \frac{q}{p}$

4. (a) $(2+5\text{i})(2-5\text{i}) = 2^2 - (5\text{i})^2 = 4-(-25) = 29$

   (b) $(3+\text{i})(3-\text{i}) = 3^2 - (\text{i})^2 = 9-(-1) = 10$

   (c) $(6+2\text{i})(6-2\text{i}) = 6^2 - (2\text{i})^2 = 36-(-4) = 40$

   (d) $(3+4\text{i})^2 = 9+24\text{i}+16\text{i}^2 = 9+24\text{i}-16 = -7+24\text{i}$

   (e) $\frac{2-3\text{i}}{3+\text{i}} = \frac{2-3\text{i}}{3+\text{i}}\times\frac{3-\text{i}}{3-\text{i}}$
   $= \frac{6-2\text{i}-9\text{i}+3\text{i}^2}{3^2-\text{i}^2}$
   $= \frac{6-11\text{i}-3}{9--1}$
   $= \frac{3-11\text{i}}{10}$
   $= 0.3-1.1\text{i}$

   (f) $\frac{3+\text{i}}{2-3\text{i}} = \frac{3+\text{i}}{2-3\text{i}}\times\frac{2+3\text{i}}{2+3\text{i}}$
   $= \frac{6+9\text{i}+2\text{i}+3\text{i}^2}{2^2-(3\text{i})^2}$
   $= \frac{6+11\text{i}-3}{4--9}$
   $= \frac{3+11\text{i}}{13}$
   $= \frac{3}{13}+\frac{11}{13}\text{i}$

5. (a) $z+w = 2-3\text{i}-3+5\text{i} = -1+2\text{i}$

   (b) $zw = (2-3\text{i})(-3+5\text{i})$
   $= -6+10\text{i}+9\text{i}-15\text{i}^2$
   $= -6+19\text{i}+15$
   $= 9+19\text{i}$

   (c) $\bar{z} = 2+3\text{i}$

   (d) $\bar{z}\bar{w} = (2+3\text{i})(-3-5\text{i})$
   $= -6-10\text{i}-9\text{i}-15\text{i}^2$
   $= -6-19\text{i}+15$
   $= 9-19\text{i}$
   Observation: $\bar{z}\bar{w} = \overline{(zw)}$

(e) $z^2 = (2-3\mathrm{i})^2$
$= 4 - 12\mathrm{i} + 9\mathrm{i}^2$
$= 4 - 12\mathrm{i} - 9$
$= -5 - 12\mathrm{i}$

(f) $(zw)^2 = (9+19\mathrm{i})^2$
$= 81 + 342\mathrm{i} + 361\mathrm{i}^2$
$= 81 + 342\mathrm{i} - 361$
$= -280 + 342\mathrm{i}$

(g) $p = \mathrm{Re}(\bar{z}) + \mathrm{Im}(\bar{w})\mathrm{i}$
$= \mathrm{Re}(z) - \mathrm{Im}(w)\mathrm{i}$
$= 2 - 5\mathrm{i}$

6. Calculator question: no working required. Refer answers in Sadler.

7. (a) $x+3=0$ or $x-2=0$
$x=-3$ $x=2$

(b) $2x-5=0$ or $x+1=0$
$2x=5$ $x=-1$
$x=2.5$

(c) $x^2-x-20=0$
$(x-5)(x+4)=0$
$x-5=0$ or $x+4=0$
$x=5$ $x=-4$

(d) $x^2-10x+24=0$
$(x-4)(x-6)=0$
$x-4=0$ or $x-6=0$
$x=4$ $x=6$

(e) $x^2-10x-24=0$
$(x-12)(x+2)=0$
$x-12=0$ or $x+2=0$
$x=12$ $x=-2$

(f) $x^2+x=12$
$x^2+x-12=0$
$(x+4)(x-3)=0$
$x+4=0$ or $x-3=0$
$x=-4$ $x=3$

8. (a) $(\mathrm{Re}(2+3\mathrm{i}))(\mathrm{Re}(5-4\mathrm{i})) = 2\times 5 = 10$

(b) $\mathrm{Re}\left((2+3\mathrm{i})(5-4\mathrm{i})\right) = \mathrm{Re}(10-8\mathrm{i}+15\mathrm{i}-12\mathrm{i}^2)$
$= \mathrm{Re}(10+7\mathrm{i}+12)$
$= \mathrm{Re}(22+7\mathrm{i})$
$= 22$

9. (a) Beginning with the right hand side:

$$\begin{aligned}
\overline{z_1+z+2} &= \overline{a+b\mathrm{i}+c+d\mathrm{i}}\\
&= \overline{(a+c)+(b+d)\mathrm{i}}\\
&= (a+c)-(b+d)\mathrm{i}\\
&= a+c-b\mathrm{i}-d\mathrm{i}\\
&= a-b\mathrm{i}+c-d\mathrm{i}\\
&= \overline{a+b\mathrm{i}}+\overline{c+d\mathrm{i}}\\
&= \bar{z}_1+\bar{z}_2\\
&= \text{L.H.S.}
\end{aligned}$$

□

(b) Beginning with the left hand side:

$$\begin{aligned}
\bar{z}_1\bar{z}_2 &= (a-b\mathrm{i})(c-d\mathrm{i})\\
&= ac-ad\mathrm{i}-bc\mathrm{i}+bd\mathrm{i}^2\\
&= ac-(ad+bc)\mathrm{i}-bd\\
&= (ac-bd)-(ad+bc)\mathrm{i}\\
&= \overline{(ac-bd)+(ad+bc)\mathrm{i}}\\
&= \overline{ac+ad\mathrm{i}-bd+bc\mathrm{i}}\\
&= \overline{ac+ad\mathrm{i}+bd\mathrm{i}^2+bc\mathrm{i}}\\
&= \overline{a(c+d\mathrm{i})+b\mathrm{i}(d\mathrm{i}+c)}\\
&= \overline{(a+b\mathrm{i})(c+d\mathrm{i})}\\
&= \overline{z_1z_2}\\
&= \text{R.H.S.}
\end{aligned}$$

□

Chapter 2

Exercise 2A

There is no need for worked solutions for any of the questions in this exercise. Refer to the answers in Sadler.

Exercise 2B

1. (a) Amplitude of $\sin(x)$ is 1.
   (b) Amplitude of $\cos(x)$ is 1, so amplitude of $2\cos(x)$ is 2.
   (c) Amplitude of $\cos(x)$ is 1, so amplitude of $4\cos(x)$ is 4.
   (d) Amplitude of $\sin(x)$ is 1, so amplitude of $-3\sin(2x)$ is 3. (Remember, amplitude can't be negative. The 2 here affects the period, not the amplitude.)
   (e) Amplitude of $\cos(x)$ is 1, so amplitude of $2\cos\left(x+\frac{\pi}{2}\right)$ is 2. (The $+\frac{\pi}{2}$ here affects the phase position, not the amplitude.)
   (f) Amplitude of $\sin(x)$ is 1, so amplitude of $-3\sin(x-\pi)$ is 3. (Remember, amplitude can't be negative. The $-\pi$ here affects the phase position, not the amplitude.)
   (g) Amplitude of $\cos(x)$ is 1, so amplitude of $5\cos(x-2)$ is 5. (The -2 here affects the phase position, not the amplitude.)
   (h) Amplitude of $\cos(x)$ is 1, so amplitude of $-3\cos(2x+\pi)$ is 3. (Amplitude can't be negative; the 2 affects period, not amplitude and the $+\pi$ affects the phase position, not the amplitude.)

2. (a) Period of $\sin x$ is $360°$.
   (b) Period of $\tan x$ is $180°$.
   (c) Period of $\sin x$ is $360°$ so the period of $2\sin x$ is also $360°$. (The 2 affects amplitude, not period.)
   (d) Period of $\sin x$ is $360°$ so the period of $\sin 2x$ is $\frac{360}{2}=180°$.
   (e) Period of $\cos x$ is $360°$ so the period of $\cos\frac{x}{2}$ is $\frac{360}{\frac{1}{2}}=720°$.
   (f) Period of $\cos x$ is $360°$ so the period of $\cos 3x$ is $\frac{360}{3}=120°$.
   (g) Period of $\tan x$ is $180°$ so the period of $3\tan 2x$ is $\frac{180}{2}=90°$. (The 3 does not affect the period.)
   (h) Period of $\sin x$ is $360°$ so the period of $3\sin\frac{x-60°}{3}$ is $\frac{360}{\frac{1}{3}}=1080°$. (The first 3 affects amplitude, not period. The $-60°$ affects phase position, not period.

3. (a) Period of $\cos x$ is 2π.
   (b) Period of $\tan x$ is π.
   (c) Period of $\cos x$ is 2π so the period of $3\cos x$ is 2π. (The 3 affects amplitude, not period.)
   (d) Period of $\cos x$ is 2π so the period of $2\cos 4x$ is $\frac{2\pi}{4}=\frac{\pi}{2}$. (The 2 affects amplitude, not period.)
   (e) The period of $\tan x$ is π so the period of $2\tan 3x$ is $\frac{\pi}{3}$. (The 2 does not affect period.)
   (f) The period of $\sin x$ is 2π so the period of $\frac{1}{2}\sin 3x$ is $\frac{2\pi}{3}$. (The $\frac{1}{2}$ affects amplitude, not period.)
   (g) The period of $\sin x$ is 2π so the period of $3\sin\frac{x}{2}$ is $\frac{2\pi}{\frac{1}{2}}=4\pi$. (The 3 affects amplitude, not period.)
   (h) The period of $2\cos 4x$ is $\frac{2\pi}{4}=\frac{\pi}{2}$. (The 2 affects amplitude, not period.)
   (i) Period of $\cos x$ is 2π so the period of $2\cos(2x-\pi)$ is$\frac{2\pi}{2}=\pi$. (The first 2 affects amplitude, not period; the $-\pi$ affects phase position, not period.
   (j) The period of $\sin x$ is 2π so the period of $2\sin 4\pi x$ is $\frac{2\pi}{4\pi}=\frac{1}{2}$. (The 2 affects amplitude, not period.)

4. (a) The maximum of $\sin x$ is 1 and occurs when $x=\frac{\pi}{2}$: coordinates $\left(\frac{\pi}{2},1\right)$
   The minimum of $\sin x$ is -1 and occurs when $x=\frac{3\pi}{2}$: coordinates $\left(\frac{3\pi}{2},-1\right)$
   (b) The "2+" increases both maximum and minimum by 2 and has no effect on when they occur. Maximum at $\left(\frac{\pi}{2},3\right)$; minimum at $\left(\frac{3\pi}{2},1\right)$.
   (c) The "$-$" has the effect of reflecting the graph of $\sin(x)$ in the $x-$axis, so the maximum becomes the minimum and vice versa. Maximum at $\left(\frac{3\pi}{2},1\right)$; minimum at $\left(\frac{\pi}{2},-1\right)$.

(d) The "2" decreases the period from 2π to π. The $x-$position of maximum and minimum is similarly halved to $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ respectively. In addition, the decreased period means that we will get two full cycles in the domain $0 \le x \le 2\pi$ so there will be two maxima and two minima, each separated by π. The "+3" means the maxima will have $y-$ values of $1+3=4$ and minima of $-1+3=2$. Thus, maxima at $\left(\frac{\pi}{4},4\right)$ and $\left(\frac{5\pi}{4},4\right)$ and minima at $\left(\frac{3\pi}{4},2\right)$ and $\left(\frac{7\pi}{4},2\right)$.

(e) The "$-\frac{\pi}{4}$" moves the graph of $\sin x$ to the right $\frac{\pi}{4}$ units so the $x-$coordinate of maximum and minimum increase to $\frac{\pi}{2}+\frac{\pi}{4}=\frac{3\pi}{4}$ and $\frac{3\pi}{2}+\frac{\pi}{4}=\frac{7\pi}{4}$. The +3 increases maximum and minimum to 4 and 2. Thus, maximum $\left(\frac{3\pi}{4},4\right)$, minimum $\left(\frac{7\pi}{4},2\right)$.

5. (a) The maximum value of $\sin x$ is 1 so the maximum value of $3\sin x$ is $3\times 1=3$. The smallest positive value of x that gives this maximum is $90°$.

   (b) The maximum value is 2 when $x = 90+30 = 120°$.

   (c) The maximum value is 2 when $x = 90-30 = 60°$.

   (d) The maximum value is 3 when $x = 270°$.

6. (a) The maximum value is 3 when $x = \frac{\pi}{2}\div 2 = \frac{\pi}{4}$.

   (b) The maximum value is 5 when $x = \frac{3\pi}{2}$.

   (c) The maximum value is 2. The maximum of $\cos x$ occurs when $x=0$ so here the maximum occurs when $x = 0-\frac{\pi}{6} = -\frac{\pi}{6}$, but this is not positive so we must add the period to get $x = -\frac{\pi}{6}+2\pi = \frac{11\pi}{6}$

   (d) The maximum value is 3 when $x = 0+\frac{\pi}{6} = \frac{\pi}{6}$

7. (a) Amplitude is 2, curve is not reflected, so $a=2$.

   (b) Amplitude is 3, curve is not reflected, so $a=3$.

   (c) Amplitude is 3, curve is reflected, so $a=-3$.

   (d) Amplitude is about 1.4, curve is reflected, so $a=-1.4$.

8. (a) Amplitude is 3, curve is not reflected, so $a=3$.

   (b) Amplitude is 2, curve is reflected, so $a=-2$.

9. (a) At $x=\frac{\pi}{4}$ the $y-$value is 2, so $a=2$.

   (b) At $x=45°$ the $y-$value is -1, so $a=-1$.

10. (a) Amplitude is 2, so $a=2$. Period is $\frac{2\pi}{3}$ which is one third the period of $\sin x$ so $b=3$.

    (b) Amplitude is 3 and curve is reflected, so $a=-3$. Period is π which is half the period of $\sin x$ so $b=2$.

    (c) Amplitude is 2, so $a=2$. Period is $60°$ which is one sixth the period of $\sin x$ so $b=6$.

    (d) Amplitude is 3, so $a=3$. Period is $90°$ which is one quarter the period of $\sin x$ so $b=4$.

11. (a) Amplitude is 1 so $a=1$. Period is π which is half the period of $\cos x$ so $b=2$.

    (b) Amplitude is 3 and the curve is reflected, so $a=-3$. Period is $\frac{2\pi}{3}$ which is one third the period of $\cos x$ so $b=3$.

    (c) Amplitude is 3 and the curve is reflected, so $a=-3$. Period is $180°$ which is half the period of $\cos x$ so $b=2$.

    (d) Amplitude is 2 so $a=2$. Period is $90°$ which is quarter the period of $\cos x$ so $b=4$.

12. (a) Amplitude is 2, so $a=2$. The broken line is $y=2\sin x$. The unbroken line is shifted to the right by $30°$ so its equation is $y=2\sin(x-30°)$ and the smallest positive value of b is $b=30$. The second smallest value of b is obtained if we consider the unbroken line as having been moved to the right by one full cycle plus $30°{=}360+30=390$.

    (b) The amplitude of 2 means $c=-2$. The solid line is the reflected sine curve shifted right by $210°$ so $d=210$.

13. (a) Amplitude is 3, period is $\frac{2\pi}{\pi}=2$.

    (b) See answers in Sadler.

14. (a) Amplitude is 5, period is $\frac{2\pi}{\pi/2}=4$.

    (b) See answers in Sadler.

15. See answers in Sadler. Curves are the same as $y=\tan x$ with a vertical dilation factor of 2. The second curve is the same shape as the first, but phase-shifted $45°$ to the left.

16. See answers in Sadler. Amplitude of curves is 3 and period is π. The second curve is the same shape as the first, but phase-shifted $\frac{\pi}{3}$ to the right.

Exercise 2C

1. $190°$ is in the 3rd quadrant where tan is **positive**.
2. $310°$ is in the 4th quadrant where cos is **positive**.
3. $-190°$ is in the 2nd quadrant wherre tan is **negative**.
4. $-170°$ is in the 3rd quadrant where sin is **negative**.
5. $555° = 360° + 195°$ so it is in the 3rd quadrant where sin is **negative**.
6. $190°$ is in the 3rd quadrant where cos is **negative**.
7. $\frac{\pi}{10}$ is in the 1st quadrant where tan is **positive**.
8. $\frac{4\pi}{5}$ is in the 2nd quadrant where sin is **positive**.
9. $\frac{\pi}{10}$ is in the 1st quadrant where cos is **positive**.
10. $-\frac{\pi}{5}$ is in the 4th quadrant where sin is **negative**.
11. $\frac{9\pi}{10}$ is in the 2nd quadrant where cos is **negative**.
12. $\frac{13\pi}{5} = 2\pi + \frac{3\pi}{5}$ so it is in the 2nd quadrant where tan is **negative**.
13. $140° = 180° - 40°$ so it makes an angle of $40°$ with the $x-$axis and is in the 2nd quadrant where sin is positive, so $\sin 140° = \sin 40°$.
14. $250° = 180° + 70°$ so it makes an angle of $70°$ with the $x-$axis and is in the 3rd quadrant where sin is negative, so $\sin 250° = -\sin 70°$.
15. $340° = 360° - 20°$ so it makes an angle of $20°$ with the $x-$axis and is in the 4th quadrant where sin is negative, so $\sin 340° = -\sin 20°$.
16. $460° = 360° + 100° = 360° + 180° - 80°$ so it makes an angle of $80°$ with the $x-$axis and is in the 2nd quadrant where sin is positive, so $\sin 460° = \sin 80°$.
17. $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$ so it makes an angle of $\frac{\pi}{6}$ with the $x-$axis and is in the 2nd quadrant where sin is positive, so $\sin\frac{5\pi}{6} = \sin\frac{\pi}{6}$.
18. $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$ so it makes an angle of $\frac{\pi}{6}$ with the $x-$axis and is in the 3rd quadrant where sin is negative, so $\sin\frac{7\pi}{6} = -\sin\frac{\pi}{6}$.
19. $\frac{11\pi}{5} = 2\pi + \frac{\pi}{5}$ so it makes an angle of $\frac{\pi}{5}$ with the $x-$axis and is in the 1st quadrant where sin is positive, so $\sin\frac{11\pi}{5} = \sin\frac{\pi}{5}$.
20. $-\frac{\pi}{5}$ makes an angle of $\frac{\pi}{5}$ with the $x-$axis and is in the 4th quadrant where sin is negative, so $\sin -\frac{\pi}{5} = -\sin\frac{\pi}{5}$.
21. $100° = 180° - 80°$ so it makes an angle of $80°$ with the $x-$axis and is in the 2nd quadrant where cos is negative, so $\cos 100° = -\cos 80°$.
22. $200° = 180° + 20°$ so it makes an angle of $20°$ with the $x-$axis and is in the 3rd quadrant where cos is negative, so $\cos 200° = -\cos 20°$.
23. $300° = 360° - 60°$ so it makes an angle of $60°$ with the $x-$axis and is in the 4th quadrant where cos is positive, so $\cos 300° = \cos 60°$.
24. $-300° = -360° + 60°$ so it makes an angle of $60°$ with the $x-$axis and is in the 1st quadrant where cos is positive, so $\cos -300° = \cos 60°$.
25. $\frac{4\pi}{5} = \pi - \frac{\pi}{5}$ so it makes an angle of $\frac{\pi}{5}$ with the $x-$axis and is in the 2nd quadrant where cos is negative, so $\cos\frac{4\pi}{5} = -\cos\frac{\pi}{5}$.
26. $\frac{9\pi}{10} = \pi - \frac{\pi}{10}$ so it makes an angle of $\frac{\pi}{10}$ with the $x-$axis and is in the 2nd quadrant where cos is negative, so $\cos\frac{9\pi}{10} = -\cos\frac{\pi}{10}$.
27. $\frac{11\pi}{10} = \pi + \frac{\pi}{10}$ so it makes an angle of $\frac{\pi}{10}$ with the $x-$axis and is in the 3rd quadrant where cos is negative, so $\cos\frac{11\pi}{10} = -\cos\frac{\pi}{10}$.
28. $\frac{21\pi}{10} = 2\pi + \frac{\pi}{10}$ so it makes an angle of $\frac{\pi}{10}$ with the $x-$axis and is in the 1st quadrant where cos is positive, so $\cos\frac{21\pi}{10} = \cos\frac{\pi}{10}$.
29. $100° = 180° - 80°$ so it makes an angle of $80°$ with the $x-$axis and is in the 2nd quadrant where tan is negative, so $\tan 100° = -\tan 80°$.
30. $200° = 180° + 20°$ so it makes an angle of $20°$ with the $x-$axis and is in the 3rd quadrant where tan is positive, so $\tan 200° = \tan 20°$.
31. $-60°$ makes an angle of $60°$ with the $x-$axis and is in the 4th quadrant where tan is negative, so $\tan -60° = -\tan 60°$.
32. $-160° = -180° + 20°$ so it makes an angle of $20°$ with the $x-$axis and is in the 2nd quadrant where tan is positive, so $\tan -160° = \tan 20°$.
33. $\frac{6\pi}{5} = \pi + \frac{\pi}{5}$ so it makes an angle of $\frac{\pi}{5}$ with the $x-$axis and is in the 3rd quadrant where tan is positive, so $\tan\frac{6\pi}{5} = \tan\frac{\pi}{5}$.
34. $-\frac{6\pi}{5} = -\pi - \frac{\pi}{5}$ so it makes an angle of $\frac{\pi}{5}$ with the $x-$axis and is in the 2nd quadrant where tan is negative, so $\tan -\frac{6\pi}{5} = -\tan\frac{\pi}{5}$.
35. $\frac{11\pi}{5} = 2\pi + \frac{\pi}{5}$ so it makes an angle of $\frac{\pi}{5}$ with the $x-$axis and is in the 1st quadrant where tan is positive, so $\tan\frac{11\pi}{5} = \tan\frac{\pi}{5}$.
36. $-\frac{21\pi}{5} = -4\pi - \frac{\pi}{5}$ so it makes an angle of $\frac{\pi}{5}$ with the $x-$axis and is in the 4th quadrant where tan is negative, so $\tan -\frac{21\pi}{5} = -\tan\frac{\pi}{5}$.
37. $300° = 360° - 60°$ and is in the 4th quadrant so
$$\begin{aligned}\sin 300° &= -\sin 60° \\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

38. $210° = 180° + 30°$ and is in the 3rd quadrant so
$$\begin{aligned}\tan 210° &= \tan 30° \\ &= \frac{1}{\sqrt{3}} \qquad \left(\text{or } \frac{\sqrt{3}}{3}\right)\end{aligned}$$

39. $240° = 180° + 60°$ and is in the 3rd quadrant so
$$\begin{aligned}\cos 240° &= -\cos 60° \\ &= -\frac{1}{2}\end{aligned}$$

40. $270° = 180° + 90°$ so $270°$ lies on the negative $y-$axis and $\cos 270° = 0$.

41. $180°$ lies on the negative $x-$axis so $\sin 180° = 0$.

42. $390° = 360° + 30°$ and is in the first quadrant so
$$\begin{aligned}\cos 390° &= \cos 30° \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

43. $-135° = -180° + 45°$ and is in the 3rd quadrant so
$$\begin{aligned}\sin -135° &= -\sin 45° \\ &= -\frac{1}{\sqrt{2}} \qquad \left(\text{or } -\frac{\sqrt{2}}{2}\right)\end{aligned}$$

44. $-135° = -180° + 45°$ and is in the 3rd quadrant so
$$\begin{aligned}\cos -135° &= -\cos 45° \\ &= -\frac{1}{\sqrt{2}} \qquad \left(\text{or } -\frac{\sqrt{2}}{2}\right)\end{aligned}$$

45. $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$ and is in the 3rd quadrant so
$$\begin{aligned}\sin \frac{7\pi}{6} &= -\sin \frac{\pi}{6} \\ &= -\frac{1}{2}\end{aligned}$$

46. $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$ and is in the 3rd quadrant so
$$\begin{aligned}\cos \frac{7\pi}{6} &= -\cos \frac{\pi}{6} \\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

47. $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$ and is in the 3rd quadrant so
$$\begin{aligned}\tan \frac{7\pi}{6} &= \tan \frac{\pi}{6} \\ &= \frac{1}{\sqrt{3}} \qquad \left(\text{or } \frac{\sqrt{3}}{3}\right)\end{aligned}$$

48. $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$ and is in the 4th quadrant so
$$\begin{aligned}\sin \frac{7\pi}{4} &= -\sin \frac{\pi}{4} \\ &= -\frac{1}{\sqrt{2}} \qquad \left(\text{or } -\frac{\sqrt{2}}{2}\right)\end{aligned}$$

49. $-\frac{7\pi}{4} = -2\pi + \frac{\pi}{4}$ and is in the 1st quadrant so
$$\begin{aligned}\cos -\frac{7\pi}{4} &= \cos \frac{\pi}{4} \\ &= \frac{1}{\sqrt{2}} \qquad \left(\text{or } \frac{\sqrt{2}}{2}\right)\end{aligned}$$

50. 6π lies on the positive $x-$axis so $\tan 6\pi = \tan 0 = 0$.

51. $\frac{5\pi}{2} = 2\pi + \frac{\pi}{2}$ so it lies on the positive $y-$axis and $\sin \frac{5\pi}{2} = \sin \frac{pi}{2} = 1$

52. $-\frac{7\pi}{3} = -2\pi - \frac{\pi}{3}$ and is in the 4th quadrant so
$$\begin{aligned}\cos -\frac{7\pi}{3} &= \cos \frac{\pi}{3} \\ &= \frac{1}{2}\end{aligned}$$

Exercise 2D

1. There will be a solution in the 1st and 4th quadrants (where cos is positive). $\cos 60° = \frac{1}{2}$ so $x = 60°$ or $x = 360 - 60 = 300°$.

2. There will be a solution in the 3rd and 4th quadrants (where sin is negative). $\sin 30° = \frac{1}{2}$ so $x = 180 + 30 = 210°$ or $x = 360 - 30 = 330°$.

3. There will be a solution in the 1st and 3rd quadrants (where tan is positive). $\tan 45° = 1$ so $x = 45°$ or $x = 180 + 45 = 225°$.

4. There will be a solution in the 3rd and 4th quadrants (where sin is negative). $\sin 45° = \frac{1}{\sqrt{2}}$ so $x = 180 + 45 = 225°$ or $x = 360 - 45 = 315°$.

5. There will be a solution in the 1st and 2nd quadrants (where sin is positive). $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ so $x = \frac{\pi}{4}$ or $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$.

6. There will be a solution in the 2nd and 3rd quadrants (where cos is negative). $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ so $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ or $x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$.

7. There will be a solution in the 2nd and 4th quadrants (where tan is negative). $\tan\frac{\pi}{4} = 1$ so $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ or $x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$.

8. There will be a solution in the 1st and 3rd quadrants (where tan is positive). $\tan\frac{\pi}{3} = \sqrt{3}$ so $x = \frac{\pi}{3}$ or $x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

9. There will be a solution in the 1st and 4th quadrants (where cos is positive). $\cos 30° = \frac{\sqrt{3}}{2}$ so $x = 30°$ or $x = -30°$.

10. There will be a solution in the 3rd and 4th quadrants (where sin is negative). $\sin 90° = 1$ so $x = -180 + 90 = -90°$ or $x = -90°$ (i.e. the same single solution).

11. There will be a solution in the 2nd and 4th quadrants (where tan is negative). $\tan 30° = \frac{1}{\sqrt{3}}$ so $x = 180 - 30 = 150°$ or $x = -30°$.

12. sin is zero for angles that fall on the $x-$axis, so $x = -180$ or $x = 0$ or $x = 180$.

13. There will be a solution in the 1st and 2nd quadrants (where sin is positive). $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ so $x = \frac{\pi}{3}$ or $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

14. There will be a solution in the 2nd and 3rd quadrants (where cos is negative). $\cos\frac{\pi}{3} = \frac{1}{2}$ so $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ or $x = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$.

15. There will be a solution in the 1st and 2nd quadrants (where sin is positive). $\sin\frac{\pi}{6} = \frac{1}{2}$ so $x = \frac{\pi}{6}$ or $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

16. cos is zero for angles that fall on the $y-$axis, so $x = \frac{\pi}{2}$ or $x = -\frac{\pi}{2}$.

17. If $0 \le x \le 180°$ then $0 \le 2x \le 360°$. $2x$ must lie in the 1st or 3rd quadrant (where tan is positive). $\tan 30° = \frac{1}{\sqrt{3}}$ so

$$\begin{aligned} 2x &= 30 &\quad\text{or}\quad 2x &= 180 + 30 = 210 \\ x &= 15° & x &= 105° \end{aligned}$$

18. If $0 \le x \le \pi$ then $0 \le 4x \le 4\pi$. $4x$ must lie in the 1st or 4th quadrant (where cos is positive). $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ so:

$$\begin{aligned} 4x &= \frac{\pi}{6} &\quad\text{or}\quad 4x &= 2\pi - \frac{\pi}{6} \\ x &= \frac{\pi}{24} & &= \frac{11\pi}{6} \\ & & x &= \frac{11\pi}{24} \end{aligned}$$

$$\begin{aligned} \text{or}\quad 4x &= 2\pi + \frac{\pi}{6} &\quad\text{or}\quad 4x &= 4\pi - \frac{\pi}{6} \\ &= \frac{13\pi}{6} & &= \frac{23\pi}{6} \\ x &= \frac{13\pi}{24} & x &= \frac{23\pi}{24} \end{aligned}$$

19. If $-90° \le x \le 90°$ then $-270° \le 3x \le 270°$. $3x$ must lie in the 1st or 2nd quadrant (where sin is positive). $\sin 30° = \frac{1}{2}$ so:

$$\begin{aligned} 3x &= -180 - 30 \quad\text{or}\quad & 3x &= 30° \quad\text{or}\quad & 3x &= 180 - 30 \\ &= -210° & x &= 10° & &= 150° \\ x &= -70° & & & x &= 50° \end{aligned}$$

20. First rearrange the equation:

$$\begin{aligned} 2\sqrt{3}\sin 2x &= 3 \\ \sin 2x &= \frac{3}{2\sqrt{3}} \\ \sin 2x &= \frac{\sqrt{3}}{2} \end{aligned}$$

If $0 \le x \le 2\pi$ then $0 \le 2x \le 4\pi$. $2x$ must lie in the 1st or 2nd quadrant (where sin is positive). $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ so:

$$\begin{aligned} 2x &= \frac{\pi}{3} &\quad\text{or}\quad 2x &= \pi - \frac{\pi}{3} \\ x &= \frac{\pi}{6} & &= \frac{2\pi}{3} \\ & & x &= \frac{\pi}{3} \end{aligned}$$

$$\begin{aligned} \text{or}\quad 2x &= 2\pi + \frac{\pi}{3} &\quad\text{or}\quad 2x &= 3\pi - \frac{\pi}{3} \\ &= \frac{7\pi}{3} & &= \frac{8\pi}{3} \\ x &= \frac{7\pi}{6} & x &= \frac{4\pi}{3} \end{aligned}$$

21. First rearrange the equation:

$$\begin{aligned} 2\cos 3x + \sqrt{3} &= 0 \\ 2\cos 3x &= -\sqrt{3} \\ \cos 3x &= -\frac{\sqrt{3}}{2} \end{aligned}$$

If $0 \le x \le 2\pi$ then $0 \le 3x \le 6\pi$. $3x$ must lie in the 2nd or 3rd quadrant (where cos is negative). $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ so:

$$\begin{aligned} 3x &= \pi - \frac{\pi}{6} &\quad\text{or}\quad 3x &= \pi + \frac{\pi}{6} \\ &= \frac{5\pi}{6} & &= \frac{7\pi}{6} \\ x &= \frac{5\pi}{18} & x &= \frac{7\pi}{18} \end{aligned}$$

$$\begin{aligned} \text{or}\quad 3x &= 3\pi - \frac{\pi}{6} &\quad\text{or}\quad 3x &= 3\pi + \frac{\pi}{6} \\ &= \frac{17\pi}{6} & &= \frac{19\pi}{6} \\ x &= \frac{17\pi}{18} & x &= \frac{19\pi}{18} \end{aligned}$$

$$\text{or} \quad 3x = 5\pi - \frac{\pi}{6} \qquad \text{or} \quad 3x = 5\pi + \frac{\pi}{6}$$
$$= \frac{29\pi}{6} \qquad = \frac{31\pi}{6}$$
$$x = \frac{29\pi}{18} \qquad x = \frac{31\pi}{18}$$

22. Using the null factor law:

$$\sin x + 1 = 0 \quad \text{or} \quad 2\sin x - 1 = 0$$
$$\sin x = -1 \qquad 2\sin x = 1$$
$$x = \frac{3\pi}{2} \qquad \sin x = \frac{1}{2}$$
$$x = \frac{\pi}{6}$$
$$\text{or} \quad x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

23. $\sin^2 x = \frac{1}{2}$

$$\sin x = \pm\frac{1}{\sqrt{2}}$$

This gives solutions in all 4 quadrants. $\sin 45° = \frac{1}{\sqrt{2}}$ so:

$$x = 45°$$
$$\text{or} \quad x = 180 - 45 = 135°$$
$$\text{or} \quad x = 180 + 45 = 225°$$
$$\text{or} \quad x = 360 - 45 = 315°$$

24. $4\cos^2 x - 3 = 0$

$$4\cos^2 x = 3$$
$$\cos^2 x = \frac{3}{4}$$
$$\cos x = \pm\frac{\sqrt{3}}{2}$$

This gives solutions in all 4 quadrants. $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ so:

$$x = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$$
$$\text{or} \quad x = -\frac{\pi}{6}$$
$$\text{or} \quad x = \frac{\pi}{6}$$
$$\text{or} \quad x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

25. $\sin x = 0$ or $2\cos x - 1 = 0$

$$x = 0 \qquad 2\cos x = 1$$
$$\text{or } x = 180° \qquad \cos x = \frac{1}{2}$$
$$\text{or } x = -180° \qquad x = 60°$$
$$\text{or } x = -60°$$

26. $\tan x = 1.5$ has solutions in the 1st and 3rd quadrant where tan is positive. $x = 0.98$ is in the 1st quadrant so there must be another solution at $x = \pi + 0.98 = 3.14 + 0.98 = 4.12$.

27. (a) $(2p-1)(p+1) = 2p^2 + 2p - p - 1$
$= 2p^2 + p - 1$

(b) By substituting $p = \cos x$ and comparing with the previous answer we see we can factorise this:

$$2\cos^2 x + \cos x - 1 = 0$$
$$(2\cos x - 1)(\cos x + 1) = 0$$

Now using the null factor law:

$$2\cos x - 1 = 0 \quad \text{or} \quad \cos x + 1 = 0$$
$$2\cos x = 1 \qquad \cos x = -1$$
$$\cos x = \frac{1}{2} \qquad x = \pi$$
$$x = \frac{\pi}{3} \qquad \text{or } x = -\pi$$
$$\text{or } x = -\frac{\pi}{3}$$

28. If $0 \le x \le 2\pi$ then $0 + \frac{\pi}{3} \le x + \frac{\pi}{3} \le 2\pi + \frac{\pi}{3}$ i.e. $\frac{\pi}{3} \le x + \frac{\pi}{3} \le \frac{7\pi}{3}$

$x + \frac{\pi}{3}$ must be in the 1st or 2nd quadrant (where sin is positive), and $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ so:

$$x + \frac{\pi}{3} = \pi - \frac{\pi}{4} \quad \text{or} \quad x + \frac{\pi}{3} = 2\pi + \frac{\pi}{4}$$
$$= \frac{3\pi}{4} \qquad = \frac{9\pi}{4}$$
$$x = \frac{3\pi}{4} - \frac{\pi}{3} \qquad x = \frac{9\pi}{4} - \frac{\pi}{3}$$
$$= \frac{9\pi}{12} - \frac{4\pi}{12} \qquad = \frac{27\pi}{12} - \frac{4\pi}{12}$$
$$= \frac{5\pi}{12} \qquad = \frac{23\pi}{12}$$

(Note: we can't use $x + \frac{\pi}{3} = \frac{\pi}{4}$ because it is outside the specified interval of possible values for x.)

Miscellaneous Exercise 2

1. $$\begin{aligned}\overrightarrow{AP} &= \frac{5}{7}\overrightarrow{AB}\\ \overrightarrow{OP} &= \overrightarrow{OA} + \frac{5}{7}(\overrightarrow{OB} - \overrightarrow{OA})\\ &= \frac{2}{7}\overrightarrow{OA} + \frac{5}{7}\overrightarrow{OB}\\ &= \frac{2}{7}(19\mathbf{i} + 18\mathbf{j}) + \frac{5}{7}(26\mathbf{i} - 17\mathbf{j})\\ &= \frac{38}{7}\mathbf{i} + \frac{36}{7}\mathbf{j} + \frac{130}{7}\mathbf{i} - \frac{85}{7}\mathbf{j}\\ &= \frac{38+130}{7}\mathbf{i} + \frac{36-85}{7}\mathbf{j}\\ &= 24\mathbf{i} - 7\mathbf{j}\\ |\overrightarrow{OP}| &= \sqrt{24^2 + 7^2}\\ &= 25 \text{ units}\end{aligned}$$

2. (a) $8^3 \times 8^4 = 8^{3+4} = 8^7$
   (b) $\sqrt{8} = 8^{\frac{1}{2}}$
   (c) $64 = 8^2$
   (d) $2 = \sqrt[3]{8} = 8^{\frac{1}{3}}$
   (e) $4 = 2^2 = \left(8^{\frac{1}{3}}\right)^2 = 8^{\frac{2}{3}}$
   (f) $0.125 = \frac{1}{8} = 8^{-1}$

3. Substitute $-3 + 7\mathrm{i}$ for z:

$$\begin{aligned}\text{L.H.S.:}\quad z^2 &= (-3+7\mathrm{i})^2\\ &= 9 - 42\mathrm{i} + 49\mathrm{i}^2\\ &= 9 - 49 - 42\mathrm{i}\\ &= -40 - 42\mathrm{i}\\ &= \text{R.H.S.}\end{aligned}$$

$\square$

It should be clear that if $z = -3+7\mathrm{i}$ is a solution then $z = -(-3+7\mathrm{i}) = 3 - 7\mathrm{i}$ is also a solution.

How would we go about finding these solutions without first being told one of them? Let the solution be $z = a + b\mathrm{i}$, with a and b real, then:

$$\begin{aligned}(a+b\mathrm{i})^2 &= -40 - 42\mathrm{i}\\ a^2 + 2ab\mathrm{i} + b^2\mathrm{i}^2 &= -40 - 42\mathrm{i}\\ a^2 - b^2 + 2ab\mathrm{i} &= -40 - 42\mathrm{i}\\ 2ab &= -42\\ ab &= -21\\ b &= -\frac{21}{a}\\ a^2 - b^2 &= -40\\ a^2 - \left(-\frac{21}{a}\right)^2 &= -40\\ a^2 - \frac{441}{a^2} &= -40\\ a^4 - 441 &= -40a^2\\ a^4 + 40a^2 - 441 &= 0\\ (a^2+49)(a^2-9) &= 0\\ a^2 &= 9\\ a &= \pm 3\\ b &= -\frac{21}{a}\\ &= \mp 7\end{aligned}$$

(We would not need to consider $a^2 + 49 = 0$ because this has no real solution and we stipulated a was real.)

4. (a) $8 = 2^3$ so $\log_2 8 = 3$
   (b) $25 = 5^2$ so $\log_5 25 = 2$
   (c) $0.2 = \frac{1}{5} = 5^{-1}$ so $\log_5 0.2 = -1$
   (d) $\sqrt{2} = 2^{\frac{1}{2}}$ so $\log_2 \sqrt{2} = \frac{1}{2}$
   (e) $1000 = 10^3$ so $\log 1000 = 3$
   (f) $a^3 \times a^7 = a^{10}$ so $\log_a(a^3 \times a^7) = 10$

5. Rearrange the equation first:

$$\begin{aligned}\sqrt{2}\sin 5x &= 1\\ \sin 5x &- \frac{1}{\sqrt{2}}\end{aligned}$$

If $0 \le x \le \pi$ then $0 \le 5x \le 5\pi$. $5x$ must be in the 1st or 2nd quadrant (where sin is positive), and $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ so:

$$5x = \frac{\pi}{4},\quad x = \frac{\pi}{20} \qquad\text{or}\qquad 5x = \pi - \frac{\pi}{4} = \frac{3\pi}{4},\quad x = \frac{3\pi}{20}$$

$$\text{or}\quad 5x = 2\pi + \frac{\pi}{4} = \frac{9\pi}{4},\quad x = \frac{9\pi}{20} \qquad\text{or}\qquad 5x = 3\pi - \frac{\pi}{4} = \frac{11\pi}{4},\quad x = \frac{11\pi}{20}$$

$$\text{or}\quad 5x = 4\pi + \frac{\pi}{4} = \frac{17\pi}{4},\quad x = \frac{17\pi}{20} \qquad\text{or}\qquad 5x = 5\pi - \frac{\pi}{4} = \frac{19\pi}{4},\quad x = \frac{19\pi}{20}$$

6. (a) $\bar{z} = -5\sqrt{2}\mathrm{i}$
   (b) $z^2 = (5\sqrt{2}\mathrm{i})^2 = 25 \times 2 \times \mathrm{i}^2 = -50$
   (c) $(1+z)^2 = 1 + 2z + z^2 = 1 + 10\sqrt{2}\mathrm{i} - 50 = -49 + 10\sqrt{2}\mathrm{i}$

7. (a) $$\begin{aligned}z + w &= 4 + 7\mathrm{i} + 2 - \mathrm{i}\\ &= 6 + 6\mathrm{i}\end{aligned}$$
   (b) $$\begin{aligned}zw &= (4+7\mathrm{i})(2-\mathrm{i})\\ &= 8 - 4\mathrm{i} + 14\mathrm{i} - 7\mathrm{i}^2\\ &= 8 + 7 + 10\mathrm{i}\\ &= 15 + 10\mathrm{i}\end{aligned}$$
   (c) $\bar{z} = 4 - 7\mathrm{i}$

(d) $$\begin{aligned}\bar{z}\bar{w} &= (4-7\text{i})(2+\text{i})\\ &= 8+4\text{i}-14\text{i}-7\text{i}^2\\ &= 8+7-10\text{i}\\ &= 15-10\text{i}\end{aligned}$$

(e) $$\begin{aligned}z^2 &= (4+7\text{i})^2\\ &= 16+56\text{i}+49\text{i}^2\\ &= 16+56\text{i}-49\\ &= -33+56\text{i}\end{aligned}$$

(f) $$\begin{aligned}(zw)^2 &= (15+10\text{i})^2\\ &= 225+300\text{i}+100\text{i}^2\\ &= 225+300\text{i}-100\\ &= 125+300\text{i}\end{aligned}$$

(g) $$\begin{aligned}p &= \text{Re}(\bar{z})+\text{Im}(\bar{w})\text{i}\\ p &= \text{Re}(z)-\text{Im}(w)\text{i}\\ &= 4+\text{i}\end{aligned}$$

8. (a) $(2, 3)$

(b) $(-5, 6)$

(c) $(0, 7)$

(d) $(3, 0)$

(e) $(3, 8) + (-2, 1) = (1, 9)$

(f) $(3, -5) + (3, 5) = (6, 0)$

(g) $(5, 3) - (2, 0) = (3, 3)$

(h) $(2, 7) - (2, -7) = (0, 14)$

(i) $(0, 2) \times (3, 5) = (0\times 3 - 2\times 5, 0\times 5 + 2\times 3) = (-10, 6)$

(j) $(-3, 1) \times (-3, -1) = ((-3)^2 + (1)^2, 0) = (10, 0)$

(k) $$\begin{aligned}(3,0)\div(2,-4) &= \frac{3}{2-4\text{i}}\times\frac{2+4\text{i}}{2+4\text{i}}\\ &= \frac{6+12\text{i}}{2^2+4^2}\\ &= \frac{6+12\text{i}}{20}\\ &= 0.3+0.6\text{i}\\ &= (0.3, 0.6)\end{aligned}$$

(l) $$\begin{aligned}(3,-8)\div(3,8) &= \frac{3-8\text{i}}{3+8\text{i}}\times\frac{3-8\text{i}}{3-8\text{i}}\\ &= \frac{3^3-48\text{i}+8^2\text{i}^2}{3^3+8^2}\\ &= \frac{9-48\text{i}-64}{9+64}\\ &= \frac{-55-48\text{i}}{73}\\ &= \left(-\frac{55}{73}, -\frac{48}{73}\right)\end{aligned}$$

9. If one solution is $x = 2 + 3\text{i}$ then
$$\begin{aligned}(2+3\text{i})^2+b(2+3\text{i})+c &= 0\\ (4+12\text{i}-9)+b(2+3\text{i})+c &= 0\\ -5+12\text{i}+2b+3b\text{i}+c &= 0\\ (-5+2b+c)+(12+3b)\text{i} &= 0\\ 12+3b &= 0\\ b &= -4\\ -5+2b+c &= 0\\ -5-8+c &= 0\\ c &= 13\\ x^2-4x+13 &= 0\end{aligned}$$

10. First factor the equation:
$$\begin{aligned}2\cos^2 x-\cos x-1 &= 0\\ (\cos x-1)(2\cos x+1) &= 0\end{aligned}$$

Now use the null factor law:
$$\begin{aligned}\cos x - 1 &= 0 &\quad\text{or}\quad 2\cos x+1 &= 0\\ \cos x &= 1 & 2\cos x &= -1\\ x &= 0 & \cos x &= -\frac{1}{2}\\ & & x &= \pi-\frac{\pi}{3}\\ & & &= \frac{2\pi}{3}\\ & & \text{or}\quad x &= -\pi+\frac{\pi}{3}\\ & & &= -\frac{2\pi}{3}\end{aligned}$$

11. $$\begin{aligned}3\sin x^\circ + 1 &= 0\\ 3\sin x^\circ &= -1\\ \sin x^\circ &= -\frac{1}{3}\end{aligned}$$
Solutions are in the 3rd and 4th quadrant where sin is negative. $\sin 19.5^\circ = \frac{1}{3}$
$$\begin{aligned}x &= 180+19.5\\ &= 199.5^\circ\\ \text{or}\quad x &= 360-19.5\\ &= 340.5^\circ\\ \text{or}\quad x &= 540+19.5\\ &= 559.5^\circ\\ \text{or}\quad x &= 720-19.5\\ &= 700.5^\circ\end{aligned}$$

12. The period is π.
$$\begin{aligned}\frac{2\pi}{a} &= \pi\\ 2\pi &= a\pi\\ a &= 2\end{aligned}$$
The solid line is phase shifted to the left $\frac{\pi}{3}$ so $b = \frac{\pi}{3}$.

Chapter 3

Exercise 3A

1. The calculator display given shows the 1st quadrant solution. There will also be a 2nd quadrant solution at $x = 180 - 14.47751219$ giving two solutions: $x \approx 14.5°$ and $x \approx 165.5°$

2. $\sin x = \pm\frac{1}{2}$ which has solutions in all four quadrants: $x = \frac{\pi}{6}$, $x = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$, $x = -\frac{\pi}{6}$, $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

3. $$\begin{aligned} 2x &= \frac{\pi}{6} & \text{or} \quad 2x &= \pi - \frac{\pi}{6} \\ x &= \frac{\pi}{12} & &= \frac{5\pi}{6} \\ & & x &= \frac{5\pi}{12} \end{aligned}$$

$$\begin{aligned} \text{or} \quad 2x &= 2\pi + \frac{\pi}{6} & \text{or} \quad 2x &= 3\pi - \frac{\pi}{6} \\ &= \frac{13\pi}{6} & &= \frac{17\pi}{6} \\ x &= \frac{13\pi}{12} & x &= \frac{17\pi}{12} \end{aligned}$$

4. $\sin^2 x + \cos^2 x = 1$ so this question simplifies to $\sin x = 1$ with solution $x = \frac{\pi}{2}$.

5. Use the null factor law:

$$\begin{aligned} 2\sin x - 1 &= 0 & \text{or} \quad \cos x &= 0 \\ 2\sin x &= 1 & x &= \frac{\pi}{2} \\ \sin x &= \frac{1}{2} & \text{or } x &= \frac{3\pi}{2} \\ x &= \frac{\pi}{6} \\ \text{or } x &= \pi - \frac{\pi}{6} \\ &= \frac{5\pi}{6} \end{aligned}$$

6. First factorise:

$$\begin{aligned} \sin x + 2\sin^2 x &= 0 \\ \sin x(1 + 2\sin x) &= 0 \end{aligned}$$

Now use the null factor law:

$$\begin{aligned} \sin x &= 0 & \text{or} \quad 1 + 2\sin x &= 0 \\ x &= 0 & 2\sin x &= -1 \\ \text{or } x &= 180° & \sin x &= -\frac{1}{2} \\ \text{or } x &= 360° & x &= 180 + 30 \\ & & &= 210° \\ & & \text{or } x &= 360 - 30 \\ & & &= 330° \end{aligned}$$

7. Use the null factor law:

$$\begin{aligned} 2\cos x + 1 &= 0 & 5\sin x - 1 &= 0 \\ 2\cos x &= -1 & 5\sin x &= 1 \\ \cos x &= -\frac{1}{2} & \sin x &= \frac{1}{5} \end{aligned}$$

The first factor will have solutions in the second and third quadrant; the second will have solutions in the first and second quadrants.

$$\begin{aligned} x &= 180 - 60 & x &= 11.5° \\ &= 120° \\ \text{or } x &= 180 + 60 & \text{or } x &= 180 - 11.5 \\ &= 240° & &= 168.5° \end{aligned}$$

8. First use the pythagorean identity to replace $\cos^2 x$ with $\sin^2 x$:

$$\begin{aligned} \sin x + \left(\sqrt{2}\right)\cos^2 x &= \sqrt{2} \\ \sin x + \left(\sqrt{2}\right)(1 - \sin^2 x) &= \sqrt{2} \\ \sin x + \sqrt{2} - \left(\sqrt{2}\right)\sin^2 x &= \sqrt{2} \\ \sin x - \left(\sqrt{2}\right)\sin^2 x &= 0 \\ \sin x\left(1 - \left(\sqrt{2}\right)\sin x\right) &= 0 \end{aligned}$$

Now use the null factor law:

$$\begin{aligned} \sin x &= 0 & \text{or} \quad 1 - \left(\sqrt{2}\right)\sin x &= 0 \\ x &= 0 & \left(\sqrt{2}\right)\sin x &= 1 \\ \text{or } x &= -\pi & \sin x &= \frac{1}{\sqrt{2}} \\ \text{or } x &= \pi & x &= \frac{\pi}{4} \\ & & \text{or } x &= \pi - \frac{\pi}{4} \\ & & &= \frac{3\pi}{4} \end{aligned}$$

9.
$$\begin{aligned}
8\sin^2 x + 4\cos^2 x &= 7\\
4\sin^2 x + (4\sin^2 x + 4\cos^2 x) &= 7\\
4\sin^2 x + 4(\sin^2 x + \cos^2 x) &= 7\\
4\sin^2 x + 4 &= 7\\
4\sin^2 x &= 3\\
\sin^2 x &= \frac{3}{4}\\
\sin x &= \pm\frac{\sqrt{3}}{2}\\
x &= \frac{\pi}{3}\\
\text{or } x &= \pi - \frac{\pi}{3}\\
&= \frac{2\pi}{3}\\
\text{or } x &= \pi + \frac{\pi}{3}\\
&= \frac{4\pi}{3}\\
\text{or } x &= 2\pi - \frac{\pi}{3}\\
&= \frac{5\pi}{3}
\end{aligned}$$

10. Rearrange and factorise:

$$\begin{aligned}
\tan^2 x + \tan x &= 2\\
\tan^2 x + \tan x - 2 &= 0\\
(\tan x + 2)(\tan x - 1) &= 0
\end{aligned}$$

Null factor law:

$$\begin{aligned}
\tan x + 2 &= 0 & \tan x - 1 &= 0\\
\tan x &= -2 & \tan x &= 1
\end{aligned}$$

The first factor has solutions in the 2nd and 4th quadrants. The second factor has solutions in the 1st and 3rd quadrants.

$$\begin{aligned}
x &= 180 - 63.4 & x &= 45^\circ\\
&= 116.6^\circ & &\\
\text{or } x &= -63.4 & x &= -180 + 45\\
& & &= -135^\circ
\end{aligned}$$

11.
$$\begin{aligned}
\sqrt{3}\sin x - 2\cos^2 x + 2 &= 0\\
\sqrt{3}\sin x - 2(1 - \sin^2 x) + 2 &= 0\\
\sqrt{3}\sin x - 2 + 2\sin^2 x + 2 &= 0\\
\sqrt{3}\sin x + 2\sin^2 x &= 0\\
\sin x\left(\sqrt{3} + 2\sin x\right) &= 0
\end{aligned}$$

Null factor law:

$$\begin{aligned}
\sin x &= 0 & \sqrt{3} + 2\sin x &= 0\\
x &= 0 & 2\sin x &= -\sqrt{3}\\
\text{or } x &= \pi & \sin x &= -\frac{\sqrt{3}}{2}\\
\text{or } x &= 2\pi & x &= \pi + \frac{\pi}{3}\\
& & &= \frac{4\pi}{3}\\
& & \text{or } x &= 2\pi - \frac{\pi}{3}\\
& & &= \frac{5\pi}{3}
\end{aligned}$$

12.
$$\begin{aligned}
5 - 4\cos x &= 4\sin^2 x\\
5 - 4\cos x &= 4(1 - \cos^2 x)\\
5 - 4\cos x &= 4 - 4\cos^2 x\\
1 - 4\cos x &= -4\cos^2 x\\
4\cos^2 x - 4\cos x + 1 &= 0\\
(2\cos x - 1)^2 &= 0\\
2\cos x - 1 &= 0\\
2\cos x &= 1\\
\cos x &= \frac{1}{2}\\
x &= \pm 60^\circ
\end{aligned}$$

13.
$$\begin{aligned}
3 &= 2\cos^2 x + 3\sin x\\
3 &= 2(1 - \sin^2 x) + 3\sin x\\
3 &= 2 - 2\sin^2 x + 3\sin x\\
2\sin^2 x - 3\sin x + 1 &= 0\\
(\sin x - 1)(2\sin x - 1) &= 0
\end{aligned}$$

$$\begin{aligned}
\sin x - 1 &= 0 & 2\sin x - 1 &= 0\\
\sin x &= 1 & 2\sin x &= 1\\
x &= \frac{\pi}{2} & \sin x &= \frac{1}{2}\\
\text{or } x &= 2\pi + \frac{\pi}{2} & x &= \frac{\pi}{6}\\
&= \frac{5\pi}{2} & \text{or } x &= \pi - \frac{\pi}{6}\\
& & &= \frac{5\pi}{6}\\
& & \text{or } x &= 2\pi + \frac{\pi}{6}\\
& & &= \frac{13\pi}{6}\\
& & \text{or } x &= 3\pi - \frac{\pi}{6}\\
& & &= \frac{17\pi}{6}
\end{aligned}$$

14.
$$
\begin{aligned}
(\sin x)(2+\sin x)+\cos^2 x &= 0\\
2\sin x+\sin^2 x+\cos^2 x &= 0\\
2\sin x+1 &= 0\\
2\sin x &= -1\\
\sin x &= -\frac{1}{2}\\
x &= \pi+\frac{\pi}{6}\\
&= \frac{7\pi}{6}\\
\text{or } x &= 2\pi-\frac{\pi}{6}\\
&= \frac{11\pi}{6}
\end{aligned}
$$

15.
$$
\begin{aligned}
2\cos x-\sqrt{3}\cos x\sin x &= 0\\
\cos x(2-\sqrt{3}\sin x &= 0
\end{aligned}
$$

$$
\begin{aligned}
\cos x &= 0 & 2-\sqrt{3}\sin x &= 0\\
x &= -\frac{\pi}{2} & \sqrt{3}\sin x &= 2\\
\text{or } x &= \frac{\pi}{2} & \sin x &= \frac{2}{\sqrt{3}}
\end{aligned}
$$

(no real solution)

16.
$$
\begin{aligned}
(\sin x)(1-\sin x) &= -\cos^2 x\\
(\sin x)(1-\sin x) &= -(1-\sin^2 x)\\
(\sin x)(1-\sin x) &= -1+\sin^2 x\\
\sin x-\sin^2 x &= -1+\sin^2 x\\
2\sin^2 x-\sin x-1 &= 0\\
(2\sin x+1)(\sin x-1) &= 0
\end{aligned}
$$

$$
\begin{aligned}
2\sin x+1 &= 0 & \sin x-1 &= 0\\
2\sin x &= -1 & \sin x &= 1\\
\sin x &= -\frac{1}{2} & x &= \frac{\pi}{2}\\
x &= \pi+\frac{\pi}{6}\\
&= \frac{7\pi}{6}
\end{aligned}
$$

17.
$$
\begin{aligned}
\sin x\tan x &= 2-\cos x\\
\sin x\frac{\sin x}{\cos x} &= 2-\cos x\\
\frac{\sin^2 x}{\cos x} &= 2-\cos x\\
\sin^2 x &= \cos x(2-\cos x)\\
\sin^2 x &= 2\cos x-\cos^2 x\\
1-\cos^2 x &= 2\cos x-\cos^2 x\\
1 &= 2\cos x\\
\cos x &= \frac{1}{2}\\
x &= \frac{\pi}{3}\\
\text{or } x &= -\frac{\pi}{3}\\
\text{or } x &= 2\pi-\frac{\pi}{3}\\
&= \frac{5\pi}{3}\text{or } x \qquad = -2\pi+\frac{\pi}{3}\\
&= \frac{-5\pi}{3}
\end{aligned}
$$

18. L.H.S.:

$$
\begin{aligned}
2\cos^2\theta+3 &= 2(1-\sin^2\theta)+3\\
&= 2-2\sin^2\theta+3\\
&= 5-2\sin^2\theta\\
&= \text{R.H.S.}
\end{aligned}
$$

□

19. L.H.S.:

$$
\begin{aligned}
\sin\theta-\cos^2\theta &= \sin\theta-(1-\sin^2\theta)\\
&= \sin\theta-1+\sin^2\theta\\
&= \sin\theta+\sin^2\theta-1\\
&= (\sin\theta)(1+\sin\theta)-1\\
&= \text{R.H.S.}
\end{aligned}
$$

□

20. L.H.S.:

$$
\begin{aligned}
(\sin\theta+\cos\theta)^2 &= \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta\\
&= 2\sin\theta\cos\theta+\sin^2\theta+\cos^2\theta\\
&= 2\sin\theta\cos\theta+1\\
&= \text{R.H.S.}
\end{aligned}
$$

□

21. R.H.S.:

$$
\begin{aligned}
(\sin\theta-\cos\theta)^2 &= \sin^2\theta-2\sin\theta\cos\theta+\cos^2\theta\\
&= \sin^2\theta+\cos^2\theta-2\sin\theta\cos\theta\\
&= 1-2\sin\theta\cos\theta\\
&= \text{L.H.S.}
\end{aligned}
$$

□

22. L.H.S.:

$$\sin^4\theta - \cos^4\theta = (\sin^2\theta + \cos^2\theta)(\sin^2\theta - \cos^2\theta)$$

(difference of perfect squares)

$$\begin{aligned}
&= 1(\sin^2\theta - \cos^2\theta)\\
&= (1-\cos^2\theta) - \cos^2\theta\\
&= 1 - 2\cos^2\theta\\
&= \text{R.H.S.}
\end{aligned}$$

□

23. L.H.S.:

$$\begin{aligned}
\sin^4\theta - \sin^2\theta &= \sin^2\theta(\sin^2\theta - 1)\\
&= (1-\cos^2\theta)(-\cos^2\theta)\\
&= -\cos^2\theta + \cos^4\theta\\
&= \cos^4\theta - \cos^2\theta\\
&= \text{R.H.S.}
\end{aligned}$$

□

24. L.H.S.:

$$\begin{aligned}
\sin^2\theta\tan^2\theta &= (1-\cos^2\theta)\tan^2\theta\\
&= \tan^2\theta - \cos^2\theta\tan^2\theta\\
&= \tan^2\theta - \cos^2\theta\frac{\sin^2\theta}{\cos^2\theta}\\
&= \tan^2\theta - \sin^2\theta\\
&= \text{R.H.S.}
\end{aligned}$$

□

25. L.H.S.:

$$\begin{aligned}
(1+\sin\theta)(1-\sin\theta) &= 1 - \sin^2\theta\\
&= \cos^2\theta\\
&= 1 + \cos^2\theta - 1\\
&= 1 + (\cos\theta + 1)(\cos\theta - 1)\\
&= \text{R.H.S.}
\end{aligned}$$

□

26. L.H.S.:

$$\begin{aligned}
\sin\theta\tan\theta + \cos\theta &= \sin\theta\frac{\sin\theta}{\cos\theta} + \cos\theta\\
&= \frac{\sin^2\theta}{\cos\theta} + \cos\theta\\
&= \frac{1-\cos^2\theta}{\cos\theta} + \cos\theta\\
&= \frac{1}{\cos\theta} - \frac{\cos^2\theta}{\cos\theta} + \cos\theta\\
&= \frac{1}{\cos\theta} - \cos\theta + \cos\theta\\
&= \frac{1}{\cos\theta}\\
&= \text{R.H.S.}
\end{aligned}$$

□

27. L.H.S.:

$$\begin{aligned}
\frac{1}{1+\tan^2\theta} &= \frac{1}{1+\frac{\sin^2\theta}{\cos^2\theta}}\\
&= \frac{1}{\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}}\\
&= \frac{1}{\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}}\\
&= \frac{\cos^2\theta}{\cos^2\theta + \sin^2\theta}\\
&= \frac{\cos^2\theta}{1}\\
&= \cos^2\theta\\
&= \text{R.H.S.}
\end{aligned}$$

□

28. R.H.S.:

$$\begin{aligned}
\frac{1+\cos\theta}{1-\cos\theta} &= \frac{1+\cos\theta}{1-\cos\theta} \times \frac{1+\cos\theta}{1+\cos\theta}\\
&= \frac{1+2\cos\theta+\cos^2\theta}{1-\cos^2\theta}\\
&= \frac{\cos^2\theta + 2\cos\theta + 1}{\sin^2\theta}\\
&= \text{L.H.S.}
\end{aligned}$$

□

29. L.H.S.:

$$\begin{aligned}
\frac{\sin\theta}{1-\cos\theta} - \frac{\cos\theta}{\sin\theta} &= \frac{\sin\theta}{1-\cos\theta} \times \frac{1+\cos\theta}{1+\cos\theta} - \frac{\cos\theta}{\sin\theta}\\
&= \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta} - \frac{\cos\theta}{\sin\theta}\\
&= \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta} - \frac{\cos\theta}{\sin\theta}\\
&= \frac{1+\cos\theta}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\\
&= \frac{1+\cos\theta-\cos\theta}{\sin\theta}\\
&= \frac{1}{\sin\theta}\\
&= \text{R.H.S.}
\end{aligned}$$

□

30. L.H.S.:

$$\begin{aligned}
\frac{1-\sin\theta\cos\theta-\cos^2\theta}{\sin^2\theta+\sin\theta\cos\theta-1} &= \frac{1-\sin\theta\cos\theta-(1-\sin^2\theta)}{(1-\cos^2\theta)+\sin\theta\cos\theta-1}\\
&= \frac{1-\sin\theta\cos\theta-1+\sin^2\theta}{1-\cos^2\theta+\sin\theta\cos\theta-1}\\
&= \frac{-\sin\theta\cos\theta+\sin^2\theta}{-\cos^2\theta+\sin\theta\cos\theta}\\
&= \frac{\sin\theta(-\cos\theta+\sin\theta)}{\cos\theta(-\cos^2\theta+\sin\theta)}\\
&= \frac{\sin\theta}{\cos\theta}\\
&= \tan\theta\\
&= \text{R.H.S.}
\end{aligned}$$

□

Exercise 3B

1. Compare this with identity ④ on page 57 of Sadler. Substitute $A = 2x$ and $B = x$:

$$\begin{aligned}\sin 2x \cos x + \cos 2x \sin x &= \sin(2x + x)\\ &= \sin 3x\end{aligned}$$

2. Compare this with identity ① on page 56 of Sadler. Substitute $A = 3x$ and $B = x$:

$$\begin{aligned}\cos 3x \cos x + \sin 3x \sin x &= \cos(3x - x)\\ &= \cos 2x\end{aligned}$$

3. Compare this with identity ③ on page 57 of Sadler. Substitute $A = 5x$ and $B = x$:

$$\begin{aligned}\sin 5x \cos x - \cos 5x \sin x &= \sin(5x - x)\\ &= \sin 4x\end{aligned}$$

4. Compare this with identity ② on page 56 of Sadler. Substitute $A = 7x$ and $B = x$:

$$\begin{aligned}\cos 7x \cos x - \sin 7x \sin x &= \cos(7x + x)\\ &= \cos 8x\end{aligned}$$

5. $$\begin{aligned}\cos 15^\circ &= \cos(45^\circ - 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}\end{aligned}$$

6. $$\begin{aligned}\tan 15^\circ &= \tan(45^\circ - 30^\circ)\\ &= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}\\ &= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \times \frac{1}{\sqrt{3}}}\\ &= \frac{\frac{\sqrt{3}}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{\frac{\sqrt{3}}{\sqrt{3}} + \frac{1}{\sqrt{3}}}\\ &= \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}} \times \frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{\sqrt{3}-1}{\sqrt{3}+1}\\ &= \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}\\ &= \frac{3 - 2\sqrt{3} + 1}{3 - 1}\\ &= \frac{4 - 2\sqrt{3}}{2}\\ &= 2 - \sqrt{3}\end{aligned}$$

7. $$\begin{aligned}\sin 75^\circ &= \sin(45^\circ + 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}\end{aligned}$$

   Compare this with the answer for question 5. We could have arrived at this more simply using the identity $\sin(90^\circ - A) = \cos(A)$ substituting $A = 15^\circ$.

8. $$\begin{aligned}\cos 75^\circ &= \cos(45^\circ + 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\end{aligned}$$

9. $$\begin{aligned}\tan 75^\circ &= \tan(45^\circ + 30^\circ)\\ &= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}\\ &= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 \times \frac{1}{\sqrt{3}}}\\ &= \frac{\frac{\sqrt{3}}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{\frac{\sqrt{3}}{\sqrt{3}} - \frac{1}{\sqrt{3}}}\\ &= \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}} \times \frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1}\\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}\\ &= \frac{3 + 2\sqrt{3} + 1}{3 - 1}\\ &= \frac{4 + 2\sqrt{3}}{2}\\ &= 2 + \sqrt{3}\end{aligned}$$

10. $$\begin{aligned}2\sin(\theta + 45^\circ) &= 2(\sin\theta \cos 45^\circ + \cos\theta \sin 45^\circ)\\ &= 2\left(\frac{1}{\sqrt{2}} \sin\theta + \frac{1}{\sqrt{2}} \cos\theta\right)\\ &= \frac{2}{\sqrt{2}} \sin\theta + \frac{2}{\sqrt{2}} \cos\theta\\ &= \sqrt{2}\sin\theta + \sqrt{2}\cos\theta\end{aligned}$$

    $a = b = \sqrt{2}$

11. $$\begin{aligned} 8\cos\left(\theta-\frac{\pi}{3}\right) &= 8\left(\cos\theta\cos\frac{\pi}{3}+\sin\theta\sin\frac{\pi}{3}\right)\\ &= 8\left(\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta\right)\\ &= 4\cos\theta+4\sqrt{3}\sin\theta\\ &= 4\sqrt{3}\sin\theta+4\cos\theta \end{aligned}$$

    $c=4\sqrt{3},\ d=4$

12. $$\begin{aligned} 4\cos(\theta+30^\circ) &= 4(\cos\theta\cos 30^\circ-\sin\theta\sin 30^\circ)\\ &= 4\left(\frac{\sqrt{3}}{2}\cos\theta-\frac{1}{2}\sin\theta\right)\\ &= 2\sqrt{3}\cos\theta-2\sin\theta \end{aligned}$$

    $e=2\sqrt{3},\ f=-2$

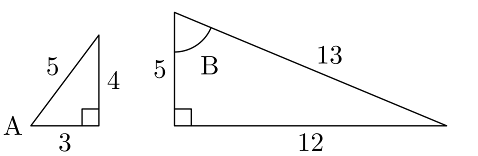
13. $$\begin{aligned} \tan(A+B) &= \frac{\tan A+\tan B}{1-\tan A\tan B}\\ &= \frac{5\sqrt{3}-\frac{\sqrt{3}}{4}}{1-5\sqrt{3}\times-\frac{\sqrt{3}}{4}}\\ &= \frac{\sqrt{3}(5-\frac{1}{4})}{1+5\times\frac{3}{4}}\\ &= \frac{\frac{19\sqrt{3}}{4}}{1+\frac{15}{4}}\\ &= \frac{\frac{19\sqrt{3}}{4}}{\frac{19}{4}}\\ &= \sqrt{3} \end{aligned}$$

    $\tan(A+B)$ is positive, so $A+B$ is in quadrant 3: $A+B=\pi+\frac{\pi}{3}=\frac{4\pi}{3}$.

14. To proceed, we need to know $\cos A$ and $\sin B$. Given that we are working with acute angles, $\cos A=\sqrt{1-\sin^2 A}$. (If the angle was not known to be acute we'd have to also consider $\cos A=-\sqrt{1-\sin^2 A}$.) This gives us

    $$\cos A=\sqrt{1-\left(\frac{4}{5}\right)} = \frac{3}{5} \qquad \sin B=\sqrt{1-\left(\frac{5}{13}\right)} = \frac{12}{13}$$

    Another way of looking at this is to think about the given values in the context of a right angle triangle. $\sin \mathrm{A}=\frac{4}{5}$ so think of a triangle with hypotenuse of 5 units and opposite of 4 units. Pythagoras' theorem gives us 3 for the other side resulting in $\cos \mathrm{A}=\frac{3}{5}$ and $\tan \mathrm{A}=\frac{4}{3}$. Similarly, given $\cos \mathrm{B}=\frac{5}{13}$ think of a triangle with hypotenuse of 13 and adjacent of 5; Pythagoras gives us 12 as the remaining side and hence $\sin \mathrm{B}=\frac{12}{13}$ and $\tan \mathrm{B}=\frac{12}{5}$.

    (a) $$\begin{aligned} \sin(\mathrm{A}+\mathrm{B}) &= \sin\mathrm{A}\cos\mathrm{B}+\cos\mathrm{A}\sin\mathrm{B}\\ &= \frac{4}{5}\times\frac{5}{13}+\frac{3}{5}\times\frac{12}{13}\\ &= \frac{20}{65}+\frac{36}{65}\\ &= \frac{56}{65} \end{aligned}$$

    (b) $$\begin{aligned} \cos(\mathrm{A}-\mathrm{B}) &= \cos\mathrm{A}\cos\mathrm{B}+\sin\mathrm{A}\sin\mathrm{B}\\ &= \frac{3}{5}\times\frac{5}{13}+\frac{4}{5}\times\frac{12}{13}\\ &= \frac{15}{65}+\frac{48}{65}\\ &= \frac{63}{65} \end{aligned}$$

15. Pythagoras (see question 14 above) gives us $\cos\mathrm{D}=\frac{24}{25}$ and $\cos\mathrm{E}=\frac{4}{5}$.

    (a) $$\begin{aligned} \sin(\mathrm{D}-\mathrm{E}) &= \sin\mathrm{D}\cos\mathrm{E}-\cos\mathrm{D}\sin\mathrm{E}\\ &= \frac{7}{25}\times\frac{4}{5}-\frac{24}{25}\times\frac{3}{5}\\ &= \frac{28}{125}-\frac{72}{125}\\ &= -\frac{44}{125} \end{aligned}$$

    (b) $$\begin{aligned} \cos(\mathrm{D}+\mathrm{E}) &= \cos\mathrm{D}\cos\mathrm{E}-\sin\mathrm{D}\sin\mathrm{E}\\ &= \frac{24}{25}\times\frac{4}{5}-\frac{7}{25}\times\frac{3}{5}\\ &= \frac{96}{125}-\frac{21}{125}\\ &= \frac{75}{125}\\ &= \frac{3}{5} \end{aligned}$$

16. To prove: $\sin\left(x+\frac{\pi}{2}\right)=\cos x$

    Proof:

    $$\text{L.H.S.:}\quad \begin{aligned} \sin\left(x+\frac{\pi}{2}\right) &= \sin x\cos\frac{\pi}{2}+\cos x\sin\frac{\pi}{2}\\ &= \sin x\times 0+\cos x\times 1\\ &= \cos x\\ &= \text{R.H.S} \end{aligned}$$

    □

17. (a) To prove: $\sin(x+2\pi)=\sin x$

    Proof:

    $$\begin{aligned} \text{L.H.S.} &= \sin(x+2\pi)\\ &= \sin x\cos 2\pi+\cos x\sin 2\pi\\ &= \sin x\times 1+\cos x\times 0\\ &= \sin x\\ &= \text{R.H.S} \end{aligned}$$

    □

(b) To prove: $\sin(x - 2\pi) = \sin x$
Proof:

$$\begin{aligned}
\text{L.H.S.} &= \sin(x - 2\pi) \\
&= \sin x \cos 2\pi - \cos x \sin 2\pi \\
&= \sin x \times 1 - \cos x \times 0 \\
&= \sin x \\
&= \text{R.H.S}
\end{aligned}$$

□

18. To prove: $\cos(x + 2\pi) = \cos x$

Proof:

$$\begin{aligned}
\text{L.H.S.} &= \cos(x + 2\pi) \\
&= \cos x \cos 2\pi - \sin x \sin 2\pi \\
&= \cos x \times 1 + \sin x \times 0 \\
&= \cos x \\
&= \text{R.H.S}
\end{aligned}$$

□

19. To prove: $\tan(x + \pi) = \tan x$

Proof:

$$\begin{aligned}
\text{L.H.S.:} \quad \tan(x + \pi) &= \frac{\tan x + \tan \pi}{1 - \tan x \tan \pi} \\
&= \frac{\tan x + 0}{1 - \tan x \times 0} \\
&= \frac{\tan x}{1} \\
&= \tan x \\
&= \text{R.H.S}
\end{aligned}$$

□

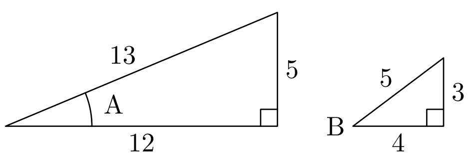
20. To prove: $\tan(-x) = -\tan x$

Proof:

$$\begin{aligned}
\text{L.H.S.:} \quad \tan(-x) &= \tan(0 - x) \\
&= \frac{\tan 0 - \tan x}{1 + \tan 0 \tan x} \\
&= \frac{-\tan x}{1 + 0 \times \tan x} \\
&= \frac{-\tan x}{1} \\
&= -\tan x \\
&= \text{R.H.S}
\end{aligned}$$

□

21. Given that they are obtuse angles, A and B fall into the 2nd quadrant so their sines are positive and cosines and tangents negative. Use Pythagoras to find the necessary ratios from those given using the triangle approach outlined in question 14 above:



θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
A	$\frac{5}{13}$	$-\frac{12}{13}$	$-\frac{5}{12}$
B	$\frac{3}{5}$	$-\frac{4}{5}$	$-\frac{3}{4}$

(a)
$$\begin{aligned}
\sin(\mathrm{A} + \mathrm{B}) &= \sin \mathrm{A} \cos \mathrm{B} + \cos \mathrm{A} \sin \mathrm{B} \\
&= \frac{5}{13} \times \left(-\frac{4}{5}\right) + \left(-\frac{12}{13}\right) \times \frac{3}{5} \\
&= -\frac{20}{65} - \frac{36}{65} \\
&= -\frac{56}{65}
\end{aligned}$$

(b)
$$\begin{aligned}
\cos(\mathrm{A} - \mathrm{B}) &= \cos \mathrm{A} \cos \mathrm{B} + \sin \mathrm{A} \sin \mathrm{B} \\
&= \left(-\frac{12}{13}\right) \times \left(-\frac{4}{5}\right) + \frac{5}{13} \times \frac{3}{5} \\
&= \frac{48}{65} + \frac{15}{65} \\
&= \frac{63}{65}
\end{aligned}$$

(c)
$$\begin{aligned}
\tan(\mathrm{A} + \mathrm{B}) &= \frac{\tan \mathrm{A} + \tan \mathrm{B}}{1 - \tan \mathrm{A} \tan \mathrm{B}} \\
&= \frac{-\frac{5}{12} - \frac{3}{4}}{1 - \left(-\frac{5}{12}\right)\left(-\frac{3}{4}\right)} \\
&= \frac{\frac{-5-9}{12}}{1 - \frac{5}{16}} \\
&= \frac{\frac{-14}{12}}{\frac{11}{16}} \\
&= -\frac{7}{6} \times \frac{16}{11} \\
&= -\frac{56}{33}
\end{aligned}$$

22. To prove: $\sin(\mathrm{A}+\mathrm{B}) - \sin(\mathrm{A}-\mathrm{B}) = 2\cos \mathrm{A} \sin \mathrm{B}$

Proof:

$$\begin{aligned}
\text{L.H.S.} &= \sin(\mathrm{A} + \mathrm{B}) - \sin(\mathrm{A} - \mathrm{B}) \\
&= \sin \mathrm{A} \cos \mathrm{B} + \cos \mathrm{A} \sin \mathrm{B} \\
&\qquad -(\sin \mathrm{A} \cos \mathrm{B} - \cos \mathrm{A} \sin \mathrm{B}) \\
&= \sin \mathrm{A} \cos \mathrm{B} + \cos \mathrm{A} \sin \mathrm{B} \\
&\qquad - \sin \mathrm{A} \cos \mathrm{B} + \cos \mathrm{A} \sin \mathrm{B} \\
&= 2 \cos \mathrm{A} \sin \mathrm{B} \\
&= \text{R.H.S}
\end{aligned}$$

□

23. To prove: $\cos(\mathrm{A}-\mathrm{B}) + \cos(\mathrm{A}+\mathrm{B}) = 2\cos \mathrm{A} \cos \mathrm{B}$

Proof:

$$\begin{aligned}
\text{L.H.S.} &= \cos(\mathrm{A} - \mathrm{B}) + \sin(\mathrm{A} + \mathrm{B}) \\
&= \cos \mathrm{A} \cos \mathrm{B} + \sin \mathrm{A} \sin \mathrm{B} \\
&\qquad + \cos \mathrm{A} \cos \mathrm{B} - \sin \mathrm{A} \sin \mathrm{B} \\
&= 2 \cos \mathrm{A} \cos \mathrm{B} \\
&= \text{R.H.S}
\end{aligned}$$

□

24. To prove: $2\cos\left(x-\frac{\pi}{6}\right)=\sin x+\sqrt{3}\cos x$

    Proof:

$$\begin{aligned}\text{L.H.S.} &= 2\cos\left(x-\frac{\pi}{6}\right)\\ &= 2\left(\cos x\cos\frac{\pi}{6}+\sin x\sin\frac{\pi}{6}\right)\\ &= 2\left(\frac{\sqrt{3}}{2}\cos x+\frac{1}{2}\sin x\right)\\ &= \sqrt{3}\cos x+\sin x\\ &= \sin x+\sqrt{3}\cos x\\ &= \text{R.H.S}\end{aligned}$$

□

25. To prove: $\tan\left(\theta+\frac{\pi}{4}\right)=\frac{1+\tan\theta}{1-\tan\theta}$

    Proof:

$$\begin{aligned}\text{L.H.S.} &= \tan\left(\theta+\frac{\pi}{4}\right)\\ &= \frac{\tan\theta+\tan\frac{\pi}{4}}{1-\tan\theta\tan\frac{\pi}{4}}\\ &= \frac{\tan\theta+1}{1-\tan\theta\times 1}\\ &= \frac{1+\tan\theta}{1-\tan\theta}\\ &= \text{R.H.S}\end{aligned}$$

□

26. To prove: $\frac{\cos(\mathrm{A}+\mathrm{B})}{\cos(\mathrm{A}-\mathrm{B})}=\frac{1-\tan\mathrm{A}\tan\mathrm{B}}{1+\tan\mathrm{A}\tan\mathrm{B}}$

    Proof:

$$\begin{aligned}\text{L.H.S.} &= \frac{\cos(\mathrm{A}+\mathrm{B})}{\cos(\mathrm{A}-\mathrm{B})}\\ &= \frac{\cos\mathrm{A}\cos\mathrm{B}-\sin\mathrm{A}\sin\mathrm{B}}{\cos\mathrm{A}\cos\mathrm{B}+\sin\mathrm{A}\sin\mathrm{B}}\\ &= \frac{\frac{\cos\mathrm{A}\cos\mathrm{B}-\sin\mathrm{A}\sin\mathrm{B}}{\cos\mathrm{A}\cos\mathrm{B}}}{\frac{\cos\mathrm{A}\cos\mathrm{B}+\sin\mathrm{A}\sin\mathrm{B}}{\cos\mathrm{A}\cos\mathrm{B}}}\\ &= \frac{1-\frac{\sin\mathrm{A}\sin\mathrm{B}}{\cos\mathrm{A}\cos\mathrm{B}}}{1+\frac{\sin\mathrm{A}\sin\mathrm{B}}{\cos\mathrm{A}\cos\mathrm{B}}}\\ &= \frac{1-\frac{\sin\mathrm{A}}{\cos\mathrm{A}}\times\frac{\sin\mathrm{B}}{\cos\mathrm{B}}}{1+\frac{\sin\mathrm{A}}{\cos\mathrm{A}}\times\frac{\sin\mathrm{B}}{\cos\mathrm{B}}}\\ &= \frac{1-\tan\mathrm{A}\tan\mathrm{B}}{1+\tan\mathrm{A}\tan\mathrm{B}}\\ &= \text{R.H.S}\end{aligned}$$

□

27. To prove:

$$\sqrt{2}(\sin x-\cos x)\sin(x+45°)=1-2\cos^2 x$$

    Proof:

$$\begin{aligned}\text{L.H.S.} &= \sqrt{2}(\sin x-\cos x)\sin(x+45°)\\ &= \sqrt{2}(\sin x-\cos x)\,(\sin x\cos 45°\\ &\qquad +\cos x\sin 45°)\\ &= \sqrt{2}(\sin x-\cos x)\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right)\\ &= (\sin x-\cos x)(\sin x+\cos x)\\ &= \sin^2 x-\cos^2 x\\ &= (1-\cos^2 x)-\cos^2 x\\ &= 1-2\cos^2 x\\ &= \text{R.H.S}\end{aligned}$$

□

28. To prove: $\tan\left(\theta+\frac{\pi}{4}\right)=\frac{1+2\sin\theta\cos\theta}{1-2\sin^2\theta}$

    Proof:

$$\begin{aligned}\text{L.H.S.} &= \tan\left(\theta+\frac{\pi}{4}\right)\\ &= \frac{\tan\theta+\tan\frac{\pi}{4}}{1-\tan\theta\tan\pi 4}\\ &= \frac{\tan\theta+1}{1-\tan\theta\times 1}\\ &= \frac{\frac{\sin\theta}{\cos\theta}+1}{1-\frac{\sin\theta}{\cos\theta}}\\ &= \frac{\frac{\sin\theta+\cos\theta}{\cos\theta}}{\frac{\cos\theta-\sin\theta}{\cos\theta}}\\ &= \frac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta}\\ &= \frac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta}\times\frac{\sin\theta+\cos\theta}{\cos\theta+\sin\theta}\\ &= \frac{\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta}{\cos^2\theta-\sin^2\theta}\\ &= \frac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta}{1-\sin^2\theta-\sin^2\theta}\\ &= \frac{1+2\sin\theta\cos\theta}{1-2\sin^2\theta}\\ &= \text{R.H.S}\end{aligned}$$

□

29. $\sin x\cos\frac{\pi}{6}+\cos x\sin\frac{\pi}{6}=\sin(x+\frac{\pi}{6})$ so the equation to solve becomes

$$\sin(x+\frac{\pi}{6})=\frac{1}{\sqrt{2}}$$

    This has solutions with $x+\frac{\pi}{6}$ in the 1st and 2nd quadrant.

$$\begin{aligned}x+\frac{\pi}{6} &= \frac{\pi}{4} & x+\frac{\pi}{6} &= \pi-\frac{\pi}{4}\\ x &= \frac{\pi}{4}-\frac{\pi}{6} & x &= \pi-\frac{\pi}{4}-\frac{\pi}{6}\\ x &= \frac{\pi}{12} & x &= \frac{7\pi}{12}\end{aligned}$$

30. $\cos x\cos 20°+\sin x\sin 20°=\cos(x-20°)$ so the equation to solve becomes

$$\cos(x-20°)=\frac{1}{2}$$

This has solutions with $x - 20^\circ$ in the 1st and 4th quadrant.

$$x - 20^\circ = 60^\circ \qquad x - 20^\circ = 360^\circ - 60^\circ$$
$$x = 80^\circ \qquad x - 20^\circ = 300^\circ$$
$$x = 320^\circ$$

31. $\sin x \cos 70^\circ + \cos x \sin 70^\circ = 0.5$

$$\sin(x + 70^\circ) = 0.5$$

This has solutions for $x + 70^\circ$ in the 1st and 2nd quadrant.

$$x + 70^\circ = 30^\circ \qquad x + 70^\circ = 180 - 30^\circ$$
$$x = -40^\circ \qquad x + 70^\circ = 150^\circ$$
$$x = 80^\circ$$

32.
$$\begin{aligned}
\sin(x + 30^\circ) &= \cos x \\
\sin x \cos 30^\circ + \cos x \sin 30^\circ &= \cos x \\
\frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x &= \cos x \\
\frac{\sqrt{3}}{2} \sin x &= \frac{1}{2} \cos x \\
\sqrt{3} \sin x &= \cos x \\
\sqrt{3} \frac{\sin x}{\cos x} &= 1 \\
\tan x &= \frac{1}{\sqrt{3}}
\end{aligned}$$

Solutions in 1st and 3rd quadrant.

$$x = 30^\circ \qquad x = 180^\circ + 30^\circ$$
$$= 210^\circ$$

Exercise 3C

1. $\cos \mathrm{A} = -\frac{4}{5}$ (by Pythagoras, and given that A is in the 2nd quadrant) and $\tan \mathrm{A} = -\frac{3}{4}$.

(a)
$$\begin{aligned}
\sin 2\mathrm{A} &= 2 \sin \mathrm{A} \cos \mathrm{A} \\
&= 2 \times \frac{3}{5} \times -\frac{4}{5} \\
&= -\frac{24}{25}
\end{aligned}$$

(b)
$$\begin{aligned}
\cos 2\mathrm{A} &= 2 \cos^2 A - 1 \\
&= 2 \times \left(\frac{4}{5}\right)^2 - 1 \\
&= \frac{7}{25}
\end{aligned}$$

(Alternatively, once you've found $\sin 2\mathrm{A}$ use Pythagoras to find $\cos 2\mathrm{A}$.)

(c)
$$\begin{aligned}
\tan 2\mathrm{A} &= \frac{2 \tan \mathrm{A}}{1 - \tan^2 A} \\
&= \frac{2 \times -\frac{3}{4}}{1 - \left(-\frac{3}{4}\right)^2} \\
&= \frac{-\frac{6}{4}}{\frac{7}{16}} \\
&= -\frac{3}{2} \times \frac{16}{7} \\
&= -\frac{24}{7}
\end{aligned}$$

2. Think of the right-angled triangle with sides 5, 12 and 13, but bear in mind that B is in the 3rd quadrant, so we have $\sin \mathrm{B} = -\frac{5}{13}$, $\cos \mathrm{B} = -\frac{12}{13}$.

(a)
$$\begin{aligned}
\sin 2\mathrm{B} &= 2 \sin \mathrm{B} \cos \mathrm{B} \\
&= 2(-\frac{5}{13})(-\frac{12}{13}) \\
&= \frac{120}{169}
\end{aligned}$$

(b)
$$\begin{aligned}
\cos 2\mathrm{B} &= 2 \cos^2 \mathrm{B} - 1 \\
&= 2(-\frac{12}{13})^2 - 1 \\
&= \frac{288 - 169}{169} \\
&= \frac{119}{169}
\end{aligned}$$

(c)
$$\begin{aligned}
\tan 2\mathrm{B} &= \frac{\sin 2\mathrm{B}}{\cos 2\mathrm{B}} \\
&= \frac{120}{119}
\end{aligned}$$

(We could use the double-angle formula for tangent, but since we have found sine and cosine this is simpler.)

3. (a) $6 \sin \mathrm{A} \cos \mathrm{A} = 3(2 \sin \mathrm{A} \cos \mathrm{A}) = 3 \sin 2\mathrm{A}$

(b)
$$\begin{aligned}
4 \sin 2\mathrm{A} \cos 2\mathrm{A} &= 2(2 \sin 2\mathrm{A} \cos 2\mathrm{A}) \\
&= 2 \sin(2 \times 2\mathrm{A}) \\
&= 2 \sin 4\mathrm{A}
\end{aligned}$$

(c)
$$\begin{aligned}
\sin \frac{\mathrm{A}}{2} \cos \frac{\mathrm{A}}{2} &= \frac{1}{2}(2 \sin \frac{\mathrm{A}}{2} \cos \frac{\mathrm{A}}{2}) \\
&= \frac{1}{2} \sin(2 \times \frac{\mathrm{A}}{2}) \\
&= \frac{1}{2} \sin \mathrm{A}
\end{aligned}$$

4. (a) $$\begin{aligned} 2\cos^2 2\mathrm{A} - 2\sin^2 2\mathrm{A} &= 2(\cos^2 2\mathrm{A} - \sin^2 2\mathrm{A}) \\ &= 2\cos(2\times 2\mathrm{A}) \\ &= 2\cos 4\mathrm{A} \end{aligned}$$

   (b) $$\begin{aligned} 1 - 2\sin^2\frac{\mathrm{A}}{2} &= \cos(2\times\frac{\mathrm{A}}{2}) \\ &= \cos\mathrm{A} \end{aligned}$$

   (c) $$\begin{aligned} 2\cos^2 2\mathrm{A} - 1 &= \cos(2\times 2\mathrm{A}) \\ &= \cos 4\mathrm{A} \end{aligned}$$

5. $\sqrt{25^2 - 24^2} = 7$. Think of the right-angled triangle with sides 7, 24 and 25, but bear in mind that θ is obtuse (so in the 2nd quadrant), so we have $\sin\theta = -\frac{5}{13}$, $\cos B = -\frac{12}{13}$.

   (a) $$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\times\frac{7}{25}\times -\frac{24}{25} \\ &= -\frac{336}{625} \end{aligned}$$

   (b) $$\begin{aligned} \cos 2\theta &= 2\cos^2\theta - 1 \\ &= 2\times\left(\frac{24}{25}\right)^2 - 1 \\ &= \frac{1152-625}{625} \\ &= \frac{527}{625} \end{aligned}$$

   (c) $$\begin{aligned} \tan 2\theta &= \frac{\sin 2\theta}{\cos 2\theta} \\ &= -\frac{336}{527} \end{aligned}$$

6. $$\begin{aligned} 4\sin x\cos x &= 1 \\ 2(2\sin x\cos x) &= 1 \\ 2\sin 2x &= 1 \\ \sin 2x &= \frac{1}{2} \end{aligned}$$

   This will have four solutions with $2x$ in 1st and 2nd quadrant.

   $$\begin{aligned} 2x &= 30^\circ \\ x &= 15^\circ \end{aligned} \qquad \text{or} \qquad \begin{aligned} 2x &= 180^\circ - 30^\circ \\ 2x &= 150^\circ \\ x &= 75^\circ \end{aligned}$$

   or

   $$\begin{aligned} 2x &= 360^\circ + 30^\circ \\ 2x &= 390^\circ \\ x &= 195 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 2x &= 540^\circ - 30^\circ \\ 2x &= 510^\circ \\ x &= 255^\circ \end{aligned}$$

7. $$\begin{aligned} \sin 2x + \cos x &= 0 \\ 2\sin x\cos x + \cos x &= 0 \\ \cos x(2\sin x + 1) &= 0 \end{aligned}$$

   The Null Factor Law gives:

   $$\begin{aligned} \cos x &= 0 \\ x &= \pm 90^\circ \end{aligned} \qquad \text{or} \qquad \begin{aligned} 2\sin x + 1 &= 0 \\ 2\sin x &= -1 \\ \sin x &= -\frac{1}{2} \\ x &= -30^\circ \\ \text{or } x &= -180^\circ + 30^\circ \\ &= -150^\circ \end{aligned}$$

8. $$\begin{aligned} 2\sin 2x - \sin x &= 0 \\ 4\sin x\cos x - \sin x &= 0 \\ \sin x(4\cos x - 1) &= 0 \end{aligned}$$

   The Null Factor Law gives:

   $$\begin{aligned} \sin x &= 0 \\ x &= 0^\circ \\ \text{or } x &= 180^\circ \\ \text{or } x &= 360^\circ \end{aligned} \qquad \text{or} \qquad \begin{aligned} 4\cos x - 1 &= 0 \\ 4\cos x &= 1 \\ \cos x &= \frac{1}{4} \\ x &= 75.5^\circ \\ \text{or } x &= 360^\circ - 75.5^\circ \\ &= 284.5^\circ \end{aligned}$$

9. $$\begin{aligned} 2\sin x\cos x &= \cos 2x \\ \sin 2x &= \cos 2x \\ \tan 2x &= 1 \end{aligned}$$

   This will have 4 solutions (since $\tan 2x$ has a period of $\frac{\pi}{2}$ and we want solutions for $0 \le x \le 2\pi$) in the 1st and 3rd quadrant.

   $$\begin{aligned} 2x &= \frac{\pi}{4} \\ x &= \frac{\pi}{8} \end{aligned} \qquad \text{or} \qquad \begin{aligned} 2x &= \pi + \frac{\pi}{4} \\ &= \frac{5\pi}{4} \\ x &= \frac{5\pi}{8} \end{aligned}$$

   $$\text{or} \qquad \begin{aligned} 2x &= 2\pi + \frac{\pi}{4} \\ &= \frac{9\pi}{4} \\ x &= \frac{9\pi}{8} \end{aligned} \qquad \text{or} \qquad \begin{aligned} 2x &= 3\pi + \frac{\pi}{4} \\ &= \frac{13\pi}{4} \\ x &= \frac{13\pi}{8} \end{aligned}$$

10. $$\begin{aligned} \cos 2x + 1 - \cos x &= 0 \\ 2\cos^2 x - 1 + 1 - \cos x &= 0 \\ 2\cos^2 x - \cos x &= 0 \\ \cos x(2\cos x - 1) &= 0 \end{aligned}$$

The Null Factor Law gives:

$$\begin{aligned} \cos x &= 0 \\ x &= \frac{\pi}{2} \\ \text{or } x &= \frac{3\pi}{2} \end{aligned} \qquad \text{or} \qquad \begin{aligned} 2\cos x - 1 &= 0 \\ 2\cos x &= 1 \\ \cos x &= \frac{1}{2} \\ x &= \frac{\pi}{3} \\ \text{or } x &= 2\pi - \frac{\pi}{3} \\ &= \frac{5\pi}{3} \end{aligned}$$

11.
$$\begin{aligned} \cos 2x + \sin x &= 0 \\ 1 - 2\sin^2 x + \sin x &= 0 \\ 2\sin^2 x - \sin x - 1 &= 0 \\ (2\sin x + 1)(\sin x - 1) &= 0 \end{aligned}$$
The Null Factor Law gives:

$$\begin{aligned} 2\sin x + 1 &= 0 \\ 2\sin x &= -1 \\ \sin x &= -\frac{1}{2} \\ x &= -\frac{\pi}{6} \\ \text{or} \quad x &= -\pi + \frac{\pi}{6} \\ &= -\frac{5\pi}{6} \end{aligned} \qquad \text{or} \qquad \begin{aligned} \sin x - 1 &= 0 \\ \sin x &= 1 \\ x &= \frac{\pi}{4} \end{aligned}$$

12. We can easily express $\sin^2 x$ and $\cos 2x$ in terms of either sine or cosine but $\cos x$ is not as readily changed to sine, so we'll go with cosine. (In this case it turns out not to really matter, but it's still a good principle.)
$$\begin{aligned} 2\sin^2 x + 5\cos x + \cos 2x &= 3 \\ 2(1 - \cos^2 x) + 5\cos x + (2\cos^2 x - 1) &= 3 \\ 2 - 2\cos^2 x + 5\cos x + 2\cos^2 x - 1 &= 3 \\ 1 + 5\cos x &= 3 \\ 5\cos x &= 2 \\ \cos x &= 0.4 \end{aligned}$$
The display given provides the prime (first quadrant) solution to this:
$$x = 66.4°$$
There will also be a solution in the 4th quadrant:
$$x = 360 - 66.4 = 293.6°$$
As we continue up to 540° we pass through the 1st quadrant again and find the third solution:
$$x = 360 + 66.4 = 426.4°$$

13. L.H.S.:
$$\begin{aligned} \sin 2\theta \tan\theta &= 2\sin\theta\cos\theta \times \frac{\sin\theta}{\cos\theta} \\ &= 2\sin^2\theta \\ &= \text{R.H.S.} \end{aligned}$$
□

14. L.H.S.:
$$\begin{aligned} \cos\theta\sin 2\theta &= \cos\theta \times 2\sin\theta\cos\theta \\ &= 2\sin\theta\cos^2\theta \\ &= 2\sin\theta(1 - \sin^2\theta) \\ &= 2\sin\theta - 2\sin^3\theta) \\ &= \text{R.H.S.} \end{aligned}$$
□

15. L.H.S.:
$$\begin{aligned} \frac{1 - \cos 2\theta}{1 + \cos 2\theta} &= \frac{1 - (1 - 2\sin^2\theta)}{1 + (2\cos^2\theta - 1)} \\ &= \frac{1 - 1 + 2\sin^2\theta}{1 + 2\cos^2\theta - 1} \\ &= \frac{2\sin^2\theta}{2\cos^2\theta} \\ &= \frac{\sin^2\theta}{\cos^2\theta} \\ &= \tan^2\theta \\ &= \text{R.H.S.} \end{aligned}$$
□

16. L.H.S.:
$$\begin{aligned} \sin\theta\tan\frac{\theta}{2} &= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \times \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \\ &= 2\sin^2\frac{\theta}{2} \\ &= 2(1 - \cos^2\frac{\theta}{2}) \\ &= 2 - 2\cos^2\frac{\theta}{2} \\ &= \text{R.H.S.} \end{aligned}$$
□

17. Although the R.H.S. looks more complicated, it will probably be easier to expand the left hand side than work out how to get a multiple angle function out of the right hand side. L.H.S.:
$$\begin{aligned} \sin 4\theta &= 2\sin 2\theta\cos 2\theta \\ &= 2(2\sin\theta\cos\theta)(\cos^2\theta - \sin^2\theta) \\ &= (4\sin\theta\cos\theta)(\cos^2\theta - \sin^2\theta) \\ &= 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta \\ &= \text{R.H.S.} \end{aligned}$$
□

18. L.H.S.:

$$\begin{aligned}\frac{\sin 2\theta - \sin\theta}{1-\cos\theta+\cos 2\theta} &= \frac{2\sin\theta\cos\theta-\sin\theta}{1-\cos\theta+(2\cos^2\theta-1)}\\ &= \frac{\sin\theta(2\cos\theta-1)}{2\cos^2\theta-\cos\theta}\\ &= \frac{\sin\theta(2\cos\theta-1)}{\cos\theta(2\cos\theta-1)}\\ &= \frac{\sin\theta}{\cos\theta}\\ &= \tan\theta\\ &= \text{R.H.S.}\end{aligned}$$

□

19. L.H.S.:

$$\begin{aligned}\cos 4\theta &= 2\cos^2 2\theta - 1\\ &= 2(\cos 2\theta)^2 - 1\\ &= 2(2\cos^2\theta-1)^2-1\\ &= 2(2\cos^2\theta-1)(2\cos^2\theta-1)-1\\ &= 2(4\cos^4\theta-4\cos^2\theta+1)-1\\ &= 8\cos^4\theta-8\cos^2\theta+2-1\\ &= 8\cos^4\theta-8\cos^2\theta+1\\ &= 1-8\cos^2\theta+8\cos^4\theta\\ &= \text{R.H.S.}\end{aligned}$$

□

Exercise 3D

1. To obtain the form $a\cos(\theta+\alpha)$ we need to rearrange our expression so it looks like the expansion of this:

$$a\cos(\theta+\alpha) = a(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$

$$\begin{aligned}\sqrt{3^2+4^2} &= 5\\ 3\cos\theta - 4\sin\theta &= 5\left(\frac{3}{5}\cos\theta - \frac{4}{5}\sin\theta\right)\\ &= 5(\cos\theta\cos\alpha - \sin\theta\sin\alpha\end{aligned}$$

giving $\cos\alpha = \frac{3}{5}$ and $\sin\alpha = \frac{4}{5}$:

$$\alpha = \cos^{-1}\frac{3}{5} = 53.1°$$

hence

$$3\cos\theta - 4\sin\theta = 5\cos(\theta+53.1°)$$

2. $$\begin{aligned}\sqrt{12^2+5^2} &= 13\\ 12\cos\theta - 5\sin\theta &= 13\left(\frac{12}{13}\cos\theta - \frac{5}{13}\sin\theta\right)\\ &= 13(\cos\theta\cos\alpha - \sin\theta\sin\alpha\end{aligned}$$

giving $\cos\alpha = \frac{12}{13}$ and $\sin\alpha = \frac{5}{13}$:

$$\alpha = \cos^{-1}\frac{12}{13} = 22.6°$$

hence

$$12\cos\theta - 5\sin\theta = 13\cos(\theta+22.6°)$$

3. To obtain the form $a\cos(\theta-\alpha)$ we need to rearrange our expression so it looks like the expansion of this:

$$a\cos(\theta-\alpha) = a(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$

$$\begin{aligned}\sqrt{4^2+3^2} &= 5\\ 4\cos\theta + 3\sin\theta &= 5\left(\frac{4}{5}\cos\theta + \frac{3}{5}\sin\theta\right)\\ &= 5(\cos\theta\cos\alpha + \sin\theta\sin\alpha)\end{aligned}$$

giving $\cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$:

$$\alpha = \cos^{-1}\frac{4}{5} = 0.64$$

hence

$$4\cos\theta + 3\sin\theta = 5\cos(\theta - 0.64)$$

4. $$\begin{aligned}\sqrt{7^2+24^2} &= 25\\ 7\cos\theta + 24\sin\theta &= 25\left(\frac{7}{25}\cos\theta + \frac{24}{25}\sin\theta\right)\\ &= 25(\cos\theta\cos\alpha + \sin\theta\sin\alpha)\end{aligned}$$

giving $\cos\alpha = \frac{7}{25}$ and $\sin\alpha = \frac{24}{25}$:

$$\alpha = \cos^{-1}\frac{7}{25} = 1.29$$

hence

$$7\cos\theta + 24\sin\theta = 25\cos(\theta - 1.29)$$

5. To obtain the form $a\sin(\theta+\alpha)$ we need to rearrange our expression so it looks like the expansion of this:

$$a\sin(\theta+\alpha) = a(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$

$$\sqrt{5^2+12^2} = 13$$
$$5\sin\theta + 12\cos\theta = 13\left(\frac{5}{13}\sin\theta + \frac{12}{13}\cos\theta\right)$$
$$= 13(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$
giving $\cos\alpha = \frac{5}{13}$ and $\sin\alpha = \frac{12}{13}$:
$$\alpha = \cos^{-1}\frac{5}{13} = 67.4°$$
hence
$$5\sin\theta + 12\cos\theta = 13\sin(\theta + 67.4°)$$

6. $$\sqrt{7^2+24^2} = 25$$
$$7\sin\theta + 24\cos\theta = 25\left(\frac{7}{25}\sin\theta + \frac{24}{25}\cos\theta\right)$$
$$= 25(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$
giving $\cos\alpha = \frac{7}{25}$ and $\sin\alpha = \frac{24}{25}$:
$$\alpha = \cos^{-1}\frac{7}{25} = 73.7°$$
hence
$$7\sin\theta + 24\cos\theta = 25\sin(\theta + 73.7°)$$

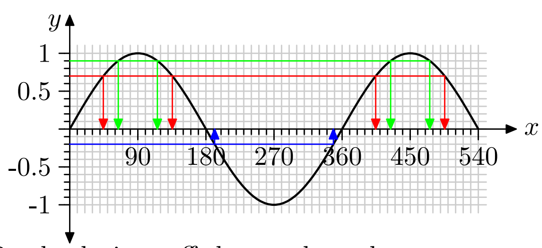
7. To obtain the form $a\sin(\theta-\alpha)$ we need to rearrange our expression so it looks like the expansion of this:
$$a\sin(\theta-\alpha) = a(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$$
$$\sqrt{4^2+3^2} = 5$$
$$4\sin\theta - 3\cos\theta = 5\left(\frac{4}{5}\sin\theta - \frac{3}{5}\cos\theta\right)$$
$$= 5(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$$
giving $\cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$:
$$\alpha = \cos^{-1}\frac{4}{5} = 0.64$$
hence
$$4\sin\theta - 3\cos\theta = 5\sin(\theta - 0.64)$$

8. $$\sqrt{2^2+3^2} = \sqrt{13}$$
$$2\sin\theta - 3\cos\theta = \sqrt{13}\left(\frac{2}{\sqrt{13}}\sin\theta - \frac{3}{\sqrt{13}}\cos\theta\right)$$
$$= \sqrt{13}(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$$
giving $\cos\alpha = \frac{2}{\sqrt{13}}$ and $\sin\alpha = \frac{3}{\sqrt{13}}$:
$$\alpha = \cos^{-1}\frac{2}{\sqrt{13}} = 0.98$$
hence
$$2\sin\theta - 3\cos\theta = \sqrt{13}\sin(\theta - 0.98)$$

9. (a) $$\sqrt{1^2+1^2} = \sqrt{2}$$
$$\cos\theta + \sin\theta = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right)$$
$$= \sqrt{2}(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$
giving $\cos\alpha = \frac{1}{\sqrt{2}}$ and $\sin\alpha = \frac{1}{\sqrt{2}}$:
$$\alpha = \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$$
hence
$$\cos\theta + \sin\theta = \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$$

(b) The maximum value of $\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$ is $\sqrt{2}$ (its amplitude) and occurs when
$$\cos\left(\theta - \frac{\pi}{4}\right) = 1$$
$$\theta - \frac{\pi}{4} = 0$$
$$\theta = \frac{\pi}{4}$$

Miscellaneous Exercise 3

1. 

Read solutions off the graph as shown.

2. Read the answer for (a) directly from the graph. For (b), read the period directly from the graph. Amplitude$=\frac{4--2}{2}=3$.

3.
$$\begin{aligned}|\mathbf{a}| &= |\mathbf{b}|\\ 1^2+p^2 &= 5^2+5^2\\ 1+p^2 &= 50\\ p^2 &= 49\\ p &= \pm 7\end{aligned}$$
$$\begin{aligned}\mathbf{c}+(2\mathbf{i}-3\mathbf{j}) &= \mathbf{a}\\ \mathbf{c} &= (\mathbf{i}\pm 7\mathbf{j})-(2\mathbf{i}-3\mathbf{j})\\ \mathbf{c} &= (1-2)\mathbf{i}+(\pm 7+3)\mathbf{j}\\ \mathbf{c} &= -\mathbf{i}+10\mathbf{j}\\ \text{or}\quad \mathbf{c} &= -\mathbf{i}-4\mathbf{j}\end{aligned}$$

4. $\sin x^\circ = -0.53$ will have solutions in the 3rd and 4th quadrants: two solutions for each full cycle of the sine graph. Thus in the interval $-360 \le x \le 360$ we will have four solutions:

- $x = -32$ (4th quadrant)
- $x = -180+32 = -148$ (3rd quadrant)
- $x = 180+32 = 212$ (3rd quadrant)
- $x = 360-32 = 328$ (4th quadrant)

5. Beginning with the Left Hand Side as most complicated (even though it is shorter):

$$\begin{aligned}\cos 3\theta &= \cos(2\theta+\theta)\\ &= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta\\ &= (2\cos^2\theta-1)\cos\theta-(2\sin\theta\cos\theta)\sin\theta\\ &= 2\cos^3\theta-\cos\theta-2\sin^2\theta\cos\theta\\ &= 2\cos^3\theta-\cos\theta-2(1-\cos^2\theta)\cos\theta\\ &= 2\cos^3\theta-\cos\theta-(2-2\cos^2\theta)\cos\theta\\ &= 2\cos^3\theta-\cos\theta-(2\cos\theta-2\cos^3\theta)\\ &= 2\cos^3\theta-\cos\theta-2\cos\theta+2\cos^3\theta\\ &= 4\cos^3\theta-3\cos\theta\\ &= \text{Right Hand Side}\end{aligned}$$

□

6.
$$\begin{aligned}\frac{1}{z} &= \frac{2+7\mathrm{i}}{1-\mathrm{i}}\\ z &= \frac{1-\mathrm{i}}{2+7\mathrm{i}}\\ &= \frac{1-\mathrm{i}}{2+7\mathrm{i}}\times\frac{2-7\mathrm{i}}{2-7\mathrm{i}}\\ &= \frac{(1-\mathrm{i})(2-7\mathrm{i})}{(2+7\mathrm{i})(2-7\mathrm{i})}\\ &= \frac{2-7\mathrm{i}-2\mathrm{i}+7\mathrm{i}^2}{2^2-(7\mathrm{i})^2}\\ &= \frac{2-9\mathrm{i}-7}{4-49\mathrm{i}^2}\\ &= \frac{-5-9\mathrm{i}}{4+49}\\ &= \frac{-5-9\mathrm{i}}{53}\\ &= -\frac{5}{53}-\frac{9}{53}\mathrm{i}\end{aligned}$$

7.
$$\begin{aligned}(a+b\mathrm{i})^2 &= 5-12\mathrm{i}\\ a^2+2ab\mathrm{i}-b^2 &= 5-12\mathrm{i}\\ a^2-b^2+2ab\mathrm{i} &= 5-12\mathrm{i}\\ 2ab &= -12\\ b &= -\frac{6}{a}\\ a^2-b^2 &= 5\\ a^2-(-\frac{6}{a})^2 &= 5\\ a^2-\frac{36}{a^2} &= 5\\ a^4-36 &= 5a^2\\ a^4-5a^2-36 &= 0\\ (a^2-9)(a^2+4) &= 0\end{aligned}$$

If we now apply the null factor law we should be able to see that $a^2+4=0$ has no real solutions, so

$$\begin{aligned}a^2-9 &= 0\\ a^2 &= 9\\ a &= \pm 3\end{aligned}$$

Now substitute back to find b:

$$\begin{aligned}b &= -\frac{6}{a}\\ &= -\frac{6}{\pm 3}\\ &= -(\pm 2)\\ &= \mp 2\end{aligned}$$

So we have either $a=3$, $b=-2$ or $a=-3$, $b=2$. (Given the square in the original problem we should have expected to get a pair of solutions like this where one is the opposite of the other.)

8. (a)
$$\begin{aligned}p+q &= 2-3\mathrm{i}+5+\mathrm{i}\\ &= 7-2\mathrm{i}\end{aligned}$$

(b)
$$\begin{aligned}p\div q &= \frac{2-3\mathrm{i}}{5+\mathrm{i}}\\ &= \frac{(2-3\mathrm{i})(5-\mathrm{i})}{(5+\mathrm{i})(5-\mathrm{i})}\\ &= \frac{10-2\mathrm{i}-15\mathrm{i}-3}{25+1}\\ &= \frac{7-17\mathrm{i}}{26}\\ &= \frac{7}{26}-\frac{17}{26}\mathrm{i}\end{aligned}$$

9. Given $z=a+b\mathrm{i}$ then $\bar{z}=a-b\mathrm{i}$.

$$\begin{aligned}z+2\bar{z} &= 9+5\mathrm{i}\\ a+b\mathrm{i}+2(a-b\mathrm{i}) &= 9+5\mathrm{i}\\ a+b\mathrm{i}+2a-2b\mathrm{i} &= 9+5\mathrm{i}\\ 3a-b\mathrm{i} &= 9+5\mathrm{i}\\ a &= 3\\ b &= -5\end{aligned}$$

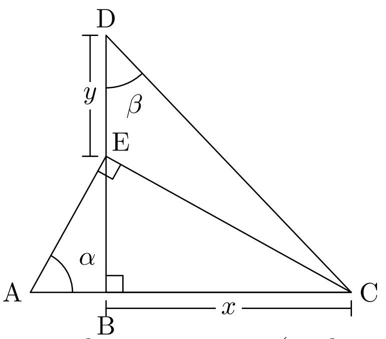
10. (a)
$$\begin{aligned} &2x^3 - 5x^2 + 8x - 3 \\ &= (px - q)(x^2 + rx + 3) \\ &= px^3 + prx^2 + 3px - qx^2 - qrx - 3q \\ &= px^3 + (pr - q)x^2 + (3p - qr)x - 3q \end{aligned}$$
Equating like terms:
$$\begin{aligned} 2x^3 &= px^3 \\ p &= 2 \\ -3 &= -3q \\ q &= 1 \\ -5x^2 &= (pr - q)x^2 \\ -5 &= pr - q \\ &= 2r - 1 \\ 2r - 1 &= -5 \\ 2r &= -4 \\ r &= -2 \\ \text{check:}\quad 8x &= (3p - qr)x \\ 3p - qr &= 3 \times 2 - 1 \times -2 \\ &= 8 \end{aligned}$$

(b) Substitute p, q, r to obtain a factorization:
$$2x^3 - 5x^2 + 8x - 3 = (2x - 1)(x^2 - 2x + 3)$$
then use the null factor law:
$$\begin{aligned} 2x - 1 &= 0 \\ x &= \frac{1}{2} \end{aligned}$$
or $\quad x^2 - 2x + 3 = 0$
$$\begin{aligned} x &= \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times 3}}{2 \times 1} \\ &= \frac{2 \pm \sqrt{-8}}{2} \\ &= \frac{2 \pm \sqrt{8}\mathrm{i}}{2} \\ &= \frac{2 \pm 2\sqrt{2}\mathrm{i}}{2} \\ x &= 1 + \sqrt{2}\mathrm{i} \\ \text{or } x &= 1 - \sqrt{2}\mathrm{i} \end{aligned}$$

11. In the 1st quadrant we obtain
$$\begin{aligned} x + \frac{\pi}{4} &= \frac{\pi}{3} \\ x &= \frac{\pi}{3} - \frac{\pi}{4} \\ &= \frac{\pi}{12} \end{aligned}$$
In the 2nd quadrant:
$$\begin{aligned} x + \frac{\pi}{4} &= \frac{2\pi}{3} \\ x &= \frac{2\pi}{3} - \frac{\pi}{4} \\ &= \frac{8\pi - 3\pi}{12} \\ &= \frac{5\pi}{12} \end{aligned}$$

12.
$$\begin{aligned} k \sin 2\theta &= 2 \sin^2 \theta \\ 2k \sin\theta \cos\theta &= 2 \sin^2 \theta \\ 2k \sin\theta \cos\theta - 2\sin^2\theta &= 0 \\ 2k \sin\theta \cos\theta - 2\sin^2\theta &= 0 \\ 2\sin\theta(k\cos\theta - \sin\theta) &= 0 \\ \sin\theta(k\cos\theta - \sin\theta) &= 0 \end{aligned}$$
$$\begin{array}{lcl} \sin\theta = 0 & \text{or} & k\cos\theta - \sin\theta = 0 \\ \theta = 0 & & k\cos\theta = \sin\theta \\ \text{or } \theta = \pi & & \sin\theta = k\cos\theta \\ \text{or } \theta = 2\pi & & \tan\theta = k \\ & & \theta = p \\ & & \text{or } \theta = \pi + p \end{array}$$

13.



$\angle$AEB is complementary to α (angle sum of a triangle).

$\angle$BEC is complementary to $\angle$AEB

$\therefore \angle\text{BEC} \cong \alpha$
$$\begin{aligned} \tan\alpha &= \frac{x}{\text{EB}} \\ \text{EB} &= \frac{x}{\tan\alpha} \\ &= \frac{x\cos\alpha}{\sin\alpha} \\ \tan\beta &= \frac{x}{y + \text{EB}} \\ y + \text{EB} &= \frac{x}{\tan\beta} \\ y &= \frac{x}{\tan\beta} - \text{EB} \\ y &= \frac{x\cos\beta}{\sin\beta} - \frac{x\cos\alpha}{\sin\alpha} \\ &= \frac{x\sin\alpha\cos\beta}{\sin\alpha\sin\beta} - \frac{x\cos\alpha\sin\beta}{\sin\alpha\sin\beta} \\ &= \frac{x\sin\alpha\cos\beta - x\cos\alpha\sin\beta}{\sin\alpha\sin\beta} \\ &= \frac{x(\sin\alpha\cos\beta - \cos\alpha\sin\beta)}{\sin\alpha\sin\beta} \\ &= \frac{x\sin(\alpha - \beta)}{\sin\alpha\sin\beta} \end{aligned}$$
□

Chapter 4

Exercise 4A

1. (a) $$\begin{aligned}\mathbf{r}_\mathrm{A} + t\mathbf{v}_\mathrm{A} &= (5\mathbf{i}+4\mathbf{j}) + t(10\mathbf{i}-\mathbf{j})\\ &= (5+10t)\mathbf{i} + (4-t)\mathbf{j}\end{aligned}$$

   (b) $$\begin{aligned}\mathbf{r}_\mathrm{B} + t\mathbf{v}_\mathrm{B} &= (6\mathbf{i}-8\mathbf{j}) + t(2\mathbf{i}+8\mathbf{j})\\ &= (6+2t)\mathbf{i} + (-8+8t)\mathbf{j}\end{aligned}$$

   (c) $$\begin{aligned}\mathbf{r}_\mathrm{C} + t\mathbf{v}_\mathrm{C} &= (2\mathbf{i}+3\mathbf{j}) + t(-4\mathbf{i}+3\mathbf{j})\\ &= (2-4t)\mathbf{i} + (3+3t)\mathbf{j}\end{aligned}$$

   (d) If the number of hours after 8am is t then the number of hours after 7am is $t+1$.

   $$\begin{aligned}&\mathbf{r}_\mathrm{D}+(t+1)\mathbf{v}_\mathrm{D}\\ &= (9\mathbf{i}-10\mathbf{j}) + (t+1)(10\mathbf{i}+6\mathbf{j})\\ &= (9\mathbf{i}-10\mathbf{j}) + t(10\mathbf{i}+6\mathbf{j}) + (10\mathbf{i}+6\mathbf{j})\\ &= (19\mathbf{i}-4\mathbf{j}) + t(10\mathbf{i}+6\mathbf{j})\\ &= (19+10t)\mathbf{i} + (-4+6t)\mathbf{j}\end{aligned}$$

   (e) If the number of hours after 8am is t then the number of hours after 9am is $t-1$.

   $$\begin{aligned}&\mathbf{r}_\mathrm{E}+(t-1)\mathbf{v}_\mathrm{E}\\ &= (16\mathbf{i}+7\mathbf{j}) + (t-1)(-4\mathbf{i}+3\mathbf{j})\\ &= (16\mathbf{i}+7\mathbf{j}) + t(-4\mathbf{i}+3\mathbf{j}) - (-4\mathbf{i}+3\mathbf{j})\\ &= (20\mathbf{i}+4\mathbf{j}) + t(-4\mathbf{i}+3\mathbf{j})\\ &= (20-4t)\mathbf{i} + (4+3t)\mathbf{j}\end{aligned}$$

   (f) If the number of hours after 8am is t then the number of hours after 8:30am is $t-0.5$.

   $$\begin{aligned}&\mathbf{r}_\mathrm{F}+(t-0.5)\mathbf{v}_\mathrm{F}\\ &= (2\mathbf{i}+3\mathbf{j}) + (t-0.5)(12\mathbf{i}-8\mathbf{j})\\ &= (2\mathbf{i}+3\mathbf{j}) + t(12\mathbf{i}-8\mathbf{j}) - (6\mathbf{i}-4\mathbf{j})\\ &= (-4\mathbf{i}+7\mathbf{j}) + t(12\mathbf{i}-8\mathbf{j})\\ &= (-4+12t)\mathbf{i} + (7-8t)\mathbf{j}\end{aligned}$$

2. (a) $$\begin{aligned}&(7\mathbf{i}+10\mathbf{j})+(6-5)(3\mathbf{i}+4\mathbf{j})\\ &= (7\mathbf{i}+10\mathbf{j}) + (3\mathbf{i}+4\mathbf{j})\\ &= (10\mathbf{i}+14\mathbf{j})\text{km}\end{aligned}$$

   (b) $$\begin{aligned}&(7\mathbf{i}+10\mathbf{j})+(7-5)(3\mathbf{i}+4\mathbf{j})\\ &= (7\mathbf{i}+10\mathbf{j}) + 2(3\mathbf{i}+4\mathbf{j})\\ &= (7\mathbf{i}+10\mathbf{j}) + (6\mathbf{i}+8\mathbf{j})\\ &= (13\mathbf{i}+18\mathbf{j})\text{km}\end{aligned}$$
   (Alternatively, add $(3\mathbf{i}+4\mathbf{i})$ to the previous answer.)

   (c) $$\begin{aligned}&(7\mathbf{i}+10\mathbf{j})+(9-5)(3\mathbf{i}+4\mathbf{j})\\ &= (7\mathbf{i}+10\mathbf{j}) + 4(3\mathbf{i}+4\mathbf{j})\\ &= (7\mathbf{i}+10\mathbf{j}) + (12\mathbf{i}+16\mathbf{j})\\ &= (19\mathbf{i}+26\mathbf{j})\text{km}\end{aligned}$$

   (d) Speed $= |3\mathbf{i}+4\mathbf{j}| = \sqrt{3^2+4^2} = 5\text{km/h}$

   (e) Ships position at 8am:

   $$\begin{aligned}&(7\mathbf{i}+10\mathbf{j})+(8-5)(3\mathbf{i}+4\mathbf{j})\\ &= (7\mathbf{i}+10\mathbf{j}) + 3(3\mathbf{i}+4\mathbf{j})\\ &= (7\mathbf{i}+10\mathbf{j}) + (9\mathbf{i}+12\mathbf{j})\\ &= (16\mathbf{i}+22\mathbf{j})\text{km}\end{aligned}$$

   Distance from lighthouse:

   $$\begin{aligned}(16\mathbf{i}+22\mathbf{j}) - (21\mathbf{i}+20\mathbf{j}) &= -5\mathbf{j}+2\mathbf{j}\\ |-5\mathbf{j}+2\mathbf{j}| &= \sqrt{(-5)^2+2^2}\\ &= \sqrt{29}\text{km}\end{aligned}$$

3. (a) Position (in km) at 9am is

   $$(9\mathbf{i}+36\mathbf{j}) - (2\mathbf{i}+12\mathbf{j}) = 7\mathbf{i}+24\mathbf{j}$$

   This represents distance from the origin of

   $$\sqrt{7^2+24^2} = 25\text{km}$$

   (b) Position (in km) at 8am is

   $$(7\mathbf{i}+24\mathbf{j}) - (2\mathbf{i}+12\mathbf{j}) = 5\mathbf{i}+12\mathbf{j}$$

   This represents distance from the origin of

   $$\sqrt{5^2+12^2} = 13\text{km}$$

4. (a) At 3pm the respective positions are

   $$\mathbf{r}_\mathrm{A} = 21\mathbf{i}+7\mathbf{j} \qquad \mathbf{r}_\mathrm{B} = 25\mathbf{i}-6\mathbf{j}$$

   The distance between them is

   $$\begin{aligned}d &= |\mathbf{r}_\mathrm{B}-\mathbf{r}_\mathrm{A}|\\ &= |(25\mathbf{i}-6\mathbf{j}) - (21\mathbf{i}+7\mathbf{j})|\\ &= |4\mathbf{i}-13\mathbf{j}|\\ &= \sqrt{4^2+(-13)^2}\\ &= \sqrt{185}\text{km}\end{aligned}$$

   (b) At 4pm the respective positions are

   $$\begin{aligned}\mathbf{r}_\mathrm{A} &= (21\mathbf{i}+7\mathbf{j}) + (10\mathbf{i}+5\mathbf{j})\\ &= 31\mathbf{i}+12\mathbf{j}\\ \mathbf{r}_\mathrm{B} &= (25\mathbf{i}-6\mathbf{j}) + (7\mathbf{i}+10\mathbf{j})\\ &= 32\mathbf{i}+4\mathbf{j}\end{aligned}$$

   The distance between them is

   $$\begin{aligned}d &= |\mathbf{r}_\mathrm{B}-\mathbf{r}_\mathrm{A}|\\ &= |(32\mathbf{i}+4\mathbf{j}) - (31\mathbf{i}+12\mathbf{j})|\\ &= |\mathbf{i}-8\mathbf{j}|\\ &= \sqrt{1^2+8^2}\\ &= \sqrt{65}\text{km}\end{aligned}$$

(c) At 5pm the respective positions are

$$\begin{aligned}\mathbf{r}_\mathrm{A} &= (21\mathbf{i}+7\mathbf{j})+2(10\mathbf{i}+5\mathbf{j})\\ &= 41\mathbf{i}+17\mathbf{j}\\ \mathbf{r}_\mathrm{B} &= (25\mathbf{i}-6\mathbf{j})+2(7\mathbf{i}+10\mathbf{j})\\ &= 39\mathbf{i}+14\mathbf{j}\end{aligned}$$

The distance between them is

$$\begin{aligned}d &= |\mathbf{r}_\mathrm{B}-\mathbf{r}_\mathrm{A}|\\ &= |(39\mathbf{i}+14\mathbf{j})-(41\mathbf{i}+17\mathbf{j})|\\ &= |-2\mathbf{i}-3\mathbf{j}|\\ &= \sqrt{(-2)^2+(-3)^2}\\ &= \sqrt{13}\text{km}\end{aligned}$$

5. (a) At 9am the respective positions are

$$\begin{aligned}\mathbf{r}_\mathrm{A} &= (-5\mathbf{i}+13\mathbf{j})+(7\mathbf{i}-2\mathbf{j})\\ &= 2\mathbf{i}+11\mathbf{j}\\ \mathbf{r}_\mathrm{B} &= (-3\mathbf{j})+(-3\mathbf{i}+2\mathbf{j})\\ &= -3\mathbf{i}-\mathbf{j}\end{aligned}$$

The distance between them is

$$\begin{aligned}d &= |\mathbf{r}_\mathrm{B}-\mathbf{r}_\mathrm{A}|\\ &= |(-3\mathbf{i}-\mathbf{j})-(2\mathbf{i}+11\mathbf{j})|\\ &= |-5\mathbf{i}-12\mathbf{j}|\\ &= \sqrt{(-5)^2+(-12)^2}\\ &= \sqrt{169}\\ &= 13\text{km}\end{aligned}$$

(b) At 10am the respective positions are

$$\begin{aligned}\mathbf{r}_\mathrm{A} &= (-5\mathbf{i}+13\mathbf{j})+2(7\mathbf{i}-2\mathbf{j})\\ &= 9\mathbf{i}+9\mathbf{j}\\ \mathbf{r}_\mathrm{B} &= (-3\mathbf{j})+2(-3\mathbf{i}+2\mathbf{j})\\ &= -6\mathbf{i}+\mathbf{j}\end{aligned}$$

The distance between them is

$$\begin{aligned}d &= |\mathbf{r}_\mathrm{B}-\mathbf{r}_\mathrm{A}|\\ &= |(-6\mathbf{i}+\mathbf{j})-(9\mathbf{i}+9\mathbf{j})|\\ &= |-15\mathbf{i}-8\mathbf{j}|\\ &= \sqrt{(-15)^2+(-8)^2}\\ &= \sqrt{289}\\ &= 17\text{km}\end{aligned}$$

6. (a) At a time t hours after 8am:

$$\begin{aligned}\mathbf{r}_\mathrm{A}(t) &= (28\mathbf{i}-5\mathbf{j})+t(-8\mathbf{i}+4\mathbf{j})\\ &= (28-8t)\mathbf{i}+(-5+4t)\mathbf{j}\\ \mathbf{r}_\mathrm{B}(t) &= (24\mathbf{j})+t(6\mathbf{i}+2\mathbf{j})\\ &= (6t)\mathbf{i}+(24+2t)\mathbf{j}\end{aligned}$$

(b) The distance between ships t hours after 8am is

$$\begin{aligned}d &= |\mathbf{r}_\mathrm{B}(t)-\mathbf{r}_\mathrm{A}(t)|\\ &= |\,((6t)\mathbf{i}+(24+2t)\mathbf{j})-\\ &\qquad((28-8t)\mathbf{i}+(-5+4t)\mathbf{j})\,|\\ &= |(-28+14t)\mathbf{i}+(29-2t)\mathbf{j}|\\ d^2 &= (-28+14t)^2+(29-2t)^2\end{aligned}$$

solving this for $d=25$:

$$\begin{aligned}25^2 &= (-28+14t)^2+(29-2t)^2\\ t &= 2 \text{ or } t=2.5\end{aligned}$$

The ships will be 25 km apart at 10am and again at 10:30am.
You should be able to solve the quadratic in the second-last line of working above manually, but provided you're confident that you can it's acceptable to use your ClassPad here. If you're unsure of the manual solution,

$$\begin{aligned}(-28+14t)^2+(29-2t)^2 &= 25^2\\ 784-784t+196t^2+841-116t+4t^2 &= 625\\ 200t^2-900t+1625 &= 625\\ 200t^2-900t+1000 &= 0\\ 2t^2-9t+10 &= 0\\ (2t-5)(t-2) &= 0\\ t &= \frac{5}{2} \text{ or } t=2\end{aligned}$$

7. The position of both ships t hours after 8am is

$$\begin{aligned}\mathbf{r}_\mathrm{A}(t) &= (12\mathbf{i}+61\mathbf{j})+t(7\mathbf{i}-8\mathbf{j})\\ &= (12+7t)\mathbf{i}+(61-8t)\mathbf{j}\\ \mathbf{r}_\mathrm{B}(t) &= (57\mathbf{i}-29\mathbf{j})+t(-2\mathbf{i}+10\mathbf{j})\\ &= (57-2t)\mathbf{i}+(-29+10t)\mathbf{j}\end{aligned}$$

Ships will collide if for some value of t:

$$\begin{aligned}\mathbf{r}_\mathrm{A}(t) &= \mathbf{r}_\mathrm{B}(t)\\ (12+7t)\mathbf{i}+(61-8t)\mathbf{j} &= (57-2t)\mathbf{i}+(-29+10t)\mathbf{j}\end{aligned}$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components separately:

$$\begin{aligned}12+7t &= 57-2t & 61-8t &= -29+10t\\ 9t &= 45 & 90 &= 18t\\ t &= 5 & t &= 5\end{aligned}$$

Thus when $t=5$ the position vectors have the same $\mathbf{i}$ components and the same $\mathbf{j}$ components: the ships collide at 1pm. The position vector of the collision is

$$(12+7\times5)\mathbf{i}+(61-8\times5)\mathbf{j} = (47\mathbf{i}+21\mathbf{j})\text{km}$$

8. The position of both ships t hours after 8am is

$$\begin{aligned}\mathbf{r}_\mathrm{A}(t) &= (-11\mathbf{i} - 8\mathbf{j}) + t(7\mathbf{i} - \mathbf{j})\\ &= (-11 + 7t)\mathbf{i} + (-8 - t)\mathbf{j}\\ \mathbf{r}_\mathrm{B}(t) &= (-2\mathbf{i} - 4\mathbf{j}) + t(4\mathbf{i} + 5\mathbf{j})\\ &= (-2 + 4t)\mathbf{i} + (-4 + 5t)\mathbf{j}\end{aligned}$$

Ships will collide if for some value of t:

$$\begin{gathered}\mathbf{r}_\mathrm{A}(t) = \mathbf{r}_\mathrm{B}(t)\\ (-11 + 7t)\mathbf{i} + (-8 - t)\mathbf{j} = (-2 + 4t)\mathbf{i} + (-4 + 5t)\mathbf{j}\end{gathered}$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components separately:

$$\begin{aligned}-11 + 7t &= -2 + 4t & -8 - t &= -4 + 5t\\ 3t &= 9 & -4 &= 6t\\ t &= 3 & t &= -\frac{2}{3}\end{aligned}$$

The position vectors have the same $\mathbf{i}$ components and the same $\mathbf{j}$ components at different times: the ships do not collide.

9. The position of both ships t hours after 8am is

$$\begin{aligned}\mathbf{r}_\mathrm{A}(t) &= (24\mathbf{i} - 25\mathbf{j}) + t(-3\mathbf{i} + 4\mathbf{j})\\ &= (24 - 3t)\mathbf{i} + (-25 + 4t)\mathbf{j}\\ \mathbf{r}_\mathrm{B}(t) &= (-9\mathbf{i} + 33\mathbf{j}) + (t - 1)(2\mathbf{i} - 5\mathbf{j})\\ &= (-9\mathbf{i} + 33\mathbf{j}) + t(2\mathbf{i} - 5\mathbf{j}) - (2\mathbf{i} - 5\mathbf{j})\\ &= (-11 + 2t)\mathbf{i} + (38 - 5t)\mathbf{j}\end{aligned}$$

Ships will collide if for some value of t:

$$\begin{gathered}\mathbf{r}_\mathrm{A}(t) = \mathbf{r}_\mathrm{B}(t)\\ (24 - 3t)\mathbf{i} + (-25 + 4t)\mathbf{j} = (-11 + 2t)\mathbf{i} + (38 - 5t)\mathbf{j}\end{gathered}$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components separately:

$$\begin{aligned}24 - 3t &= -11 + 2t & -25 + 4t &= 38 - 5t\\ 35 &= 5t & 9t &= 63\\ t &= 7 & t &= 7\end{aligned}$$

Thus when $t = 7$ the position vectors have the same $\mathbf{i}$ components and the same $\mathbf{j}$ components: the ships collide at 3pm. The position vector of the collision is

$$(24 - 3 \times 7)\mathbf{i} + (-25 + 4 \times 7)\mathbf{j} = (3\mathbf{i} + 3\mathbf{j})\text{km}$$

10. The position of both ships t hours after 9am is

$$\begin{aligned}\mathbf{r}_\mathrm{A}(t) &= (-6\mathbf{i} + 44\mathbf{j}) + (t - 0.5)(4\mathbf{i} - 6\mathbf{j})\\ &= (-6\mathbf{i} + 44\mathbf{j}) + t(4\mathbf{i} - 6\mathbf{j}) - (2\mathbf{i} - 3\mathbf{j})\\ &= (-8 + 4t)\mathbf{i} + (47 - 6t)\mathbf{j}\\ \mathbf{r}_\mathrm{B}(t) &= (2\mathbf{i} - 18\mathbf{j}) + t(2\mathbf{i} + 7\mathbf{j})\\ &= (2 + 2t)\mathbf{i} + (-18 + 7t)\mathbf{j}\end{aligned}$$

Ships will collide if for some value of t:

$$\begin{gathered}\mathbf{r}_\mathrm{A}(t) = \mathbf{r}_\mathrm{B}(t)\\ (-8 + 4t)\mathbf{i} + (47 - 6t)\mathbf{j} = (2 + 2t)\mathbf{i} + (-18 + 7t)\mathbf{j}\end{gathered}$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components separately:

$$\begin{aligned}-8 + 4t &= 2 + 2t & 47 - 6t &= -18 + 7t\\ 2t &= 10 & 65 &= 13t\\ t &= 5 & t &= 5\end{aligned}$$

Thus when $t = 5$ the position vectors have the same $\mathbf{i}$ components and the same $\mathbf{j}$ components: the ships collide at 2pm. The position vector of the collision is

$$(-8 + 4 \times 5)\mathbf{i} + (47 - 6 \times 5)\mathbf{j} = (12\mathbf{i} + 17\mathbf{j})\text{km}$$

11. The position of both ships t hours after noon is

$$\begin{aligned}\mathbf{r}_\mathrm{A}(t) &= (-11\mathbf{i} + 4\mathbf{j}) + t(10\mathbf{i} - 4\mathbf{j})\\ &= (-11 + 10t)\mathbf{i} + (4 - 4t)\mathbf{j}\\ \mathbf{r}_\mathrm{B}(t) &= (3\mathbf{i} - 5\mathbf{j}) + (t - 0.5)(7\mathbf{i} + 5\mathbf{j})\\ &= (3\mathbf{i} - 5\mathbf{j}) + t(7\mathbf{i} + 5\mathbf{j}) - (3.5\mathbf{i} + 2.5\mathbf{j})\\ &= (-0.5 + 7t)\mathbf{i} + (-7.5 + 5t)\mathbf{j}\end{aligned}$$

Ships will collide if for some value of t:

$$\begin{gathered}\mathbf{r}_\mathrm{A}(t) = \mathbf{r}_\mathrm{B}(t)\\ (-11 + 10t)\mathbf{i} + (4 - 4t)\mathbf{j} = (-0.5 + 7t)\mathbf{i} + (-7.5 + 5t)\mathbf{j}\end{gathered}$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components separately:

$$\begin{aligned}-11 + 10t &= -0.5 + 7t & 4 - 4t &= -7.5 + 5t\\ 3t &= 10.5 & 11.5 &= 9t\\ t &= 3.5 & t &= 1.25\end{aligned}$$

The position vectors have the same $\mathbf{i}$ components and the same $\mathbf{j}$ components at different times: the ships do not collide.

12. (a) The position of the three ships t hours after 8am is

$$\begin{aligned}\mathbf{r}_\mathrm{P}(t) &= (-23\mathbf{i} + 3\mathbf{j}) + t(18\mathbf{i} + 4\mathbf{j})\\ &= (-23 + 18t)\mathbf{i} + (3 + 4t)\mathbf{j}\\ \mathbf{r}_\mathrm{Q}(t) &= (7\mathbf{i} + 30\mathbf{j}) + t(12\mathbf{i} - 10\mathbf{j})\\ &= (7 + 12t)\mathbf{i} + (30 - 10t)\mathbf{j}\\ \mathbf{r}_\mathrm{R}(t) &= (32\mathbf{i} - 30\mathbf{j}) + t(2\mathbf{i} + 14\mathbf{j})\\ &= (32 + 2t)\mathbf{i} + (-30 + 14t)\mathbf{j}\end{aligned}$$

Ships P and Q will collide if for some value of t:

$$\begin{gathered}\mathbf{r}_\mathrm{P}(t) = \mathbf{r}_\mathrm{Q}(t)\\ (-23 + 18t)\mathbf{i} + (3 + 4t)\mathbf{j} = (7 + 12t)\mathbf{i} + (30 - 10t)\mathbf{j}\end{gathered}$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components separately:

$$\begin{aligned}-23 + 18t &= 7 + 12t & 3 + 4t &= 30 - 10t\\ 6t &= 30 & 14t &= 27\\ t &= 5 & t &= \frac{27}{14}\end{aligned}$$

The position vectors have the same **i** components and the same **j** components at different times: ships P and Q do not collide.
Ships P and R will collide if for some value of t:

$$\mathbf{r}_\mathrm{P}(t) = \mathbf{r}_\mathrm{R}(t)$$
$$(-23 + 18t)\mathbf{i} + (3 + 4t)\mathbf{j} = (32 + 2t)\mathbf{i} + (-30 + 14t)\mathbf{j}$$

Equating **i** and **j** components separately:

$$\begin{aligned} -23 + 18t &= 32 + 2t & 3 + 4t &= -30 + 14t \\ 16t &= 55 & 33 &= 10t \\ t &= \frac{55}{16} & t &= \frac{33}{10} \end{aligned}$$

The position vectors have the same **i** components and the same **j** components at different times: ships P and R do not collide.
Ships Q and R will collide if for some value of t:

$$\mathbf{r}_\mathrm{Q}(t) = \mathbf{r}_\mathrm{R}(t)$$
$$(7 + 12t)\mathbf{i} + (30 - 10t)\mathbf{j} = (32 + 2t)\mathbf{i} + (-30 + 14t)\mathbf{j}$$

Equating **i** and **j** components separately:

$$\begin{aligned} 7 + 12t &= 32 + 2t & 30 - 10t &= -30 + 14t \\ 10t &= 25 & 60 &= 24t \\ t &= \frac{5}{2} & t &= \frac{5}{2} \end{aligned}$$

Ships Q and R collide at 10:30am.
The position of ships Q and R at the collision is:

$$(7+12\times2.5)\mathbf{i}+(30-10\times2.5)\mathbf{j} = (37\mathbf{i}+5\mathbf{j})\text{km}$$

(b) The position of ship P at 10:30am is

$$\begin{aligned} &(-23 + 18 \times 2.5)\mathbf{i} + (3 + 4 \times 2.5)\mathbf{j} \\ &\qquad = (22\mathbf{i} + 13\mathbf{j})\text{km} \end{aligned}$$

Distance from the collision is

$$\begin{aligned} d &= \sqrt{(37-22)^2 + (5-13)^2} \\ &= \sqrt{15^2 + (-8)^2} \\ &= \sqrt{225 + 64} \\ &= \sqrt{289} \\ &= 17\text{km} \end{aligned}$$

13. At 10:05am when *Brig* starts moving her position is $(-2\mathbf{i} + 7\mathbf{j})$km. Her destination is the location of *Ajax* at 10:45am:

$$(2\mathbf{i} + 12\mathbf{j}) + \frac{45}{60}(8\mathbf{i} - 4\mathbf{j}) = (8\mathbf{j} + 9\mathbf{j})\text{km}$$

She needs to travel a total displacement of

$$(8\mathbf{i} + 9\mathbf{j}) - (-2\mathbf{i} + 7\mathbf{j}) = (10\mathbf{i} + 2\mathbf{j})\text{km}$$

To do this in the 40 minutes between 10:05 and 10:45 she needs to have velocity

$$\begin{aligned} v &= (10\mathbf{i} + 2\mathbf{j}) \div \frac{40}{60} \\ &= (10\mathbf{i} + 2\mathbf{j}) \times \frac{3}{2} \\ &= (15\mathbf{i} + 3\mathbf{j})\text{km/h} \end{aligned}$$

Exercise 4B

Note that the answers given where a vector or parametric equation of a line is requested are not uniquely correct. You may have correct answers that look different from those given here or in Sadler. For example,

$$(2 + 5\lambda)\mathbf{i} + (3 - \lambda)\mathbf{j}$$

and

$$(-3 - 5\lambda)\mathbf{i} + (4 + \lambda)\mathbf{j}$$

and

$$(7 + 10\lambda)\mathbf{i} + (2 - 2\lambda)\mathbf{j}$$

all represent the same line. In each the parameter λ means something different, so the same value of λ will give different points, but every point on one is also a point on the others, just for different values of λ.

1.
$$\begin{aligned} \mathbf{r} &= \mathbf{a} + \lambda\mathbf{b} \\ &= 2\mathbf{i} + 3\mathbf{j} + \lambda(5\mathbf{i} - \mathbf{j}) \\ &= (2 + 5\lambda)\mathbf{i} + (3 - \lambda)\mathbf{j} \end{aligned}$$

2.
$$\begin{aligned} \mathbf{r} &= \mathbf{a} + \lambda\mathbf{b} \\ &= 3\mathbf{i} - 2\mathbf{j} + \lambda(\mathbf{i} + \mathbf{j}) \\ &= (3 + \lambda)\mathbf{i} + (-2 + \lambda)\mathbf{j} \end{aligned}$$

3.
$$\begin{aligned} \mathbf{r} &= \mathbf{a} + \lambda\mathbf{b} \\ &= 5\mathbf{i} + 3\mathbf{j} + \lambda(-2\mathbf{j}) \\ &= 5\mathbf{i} + (3 - 2\lambda)\mathbf{j} \end{aligned}$$

4.
$$\begin{aligned} \mathbf{r} &= \mathbf{a} + \lambda\mathbf{b} \\ &= 5\mathbf{j} + \lambda(3\mathbf{i} - 10\mathbf{j}) \\ &= 3\lambda\mathbf{i} + (5 - 10\lambda)\mathbf{j} \end{aligned}$$

5.
$$\begin{aligned}\mathbf{r} &= \mathbf{a} + \lambda\mathbf{b}\\ &= \begin{pmatrix}2\\-3\end{pmatrix} + \lambda\begin{pmatrix}1\\4\end{pmatrix}\\ &= \begin{pmatrix}2+\lambda\\-3+4\lambda\end{pmatrix}\end{aligned}$$

6.
$$\begin{aligned}\mathbf{r} &= \mathbf{a} + \lambda\mathbf{b}\\ &= \begin{pmatrix}0\\5\end{pmatrix} + \lambda\begin{pmatrix}5\\0\end{pmatrix}\\ &= \begin{pmatrix}5\lambda\\5\end{pmatrix}\end{aligned}$$

7.
$$\begin{aligned}\mathbf{r} &= \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a})\\ &= 5\mathbf{i} + 3\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j} - 5\mathbf{i} - 3\mathbf{j})\\ &= 5\mathbf{i} + 3\mathbf{j} + \lambda(-3\mathbf{i} - 4\mathbf{j})\\ &= (5-3\lambda)\mathbf{i} + (3-4\lambda)\mathbf{j}\end{aligned}$$

8.
$$\begin{aligned}\mathbf{r} &= \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a})\\ &= 6\mathbf{i} + 7\mathbf{j} + \lambda(-5\mathbf{i} + 2\mathbf{j} - 6\mathbf{i} - 7\mathbf{j})\\ &= 6\mathbf{i} + 7\mathbf{j} + \lambda(-11\mathbf{i} - 5\mathbf{j})\\ &= (6-11\lambda)\mathbf{i} + (7-5\lambda)\mathbf{j}\end{aligned}$$

9.
$$\begin{aligned}\mathbf{r} &= \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a})\\ &= \begin{pmatrix}-6\\3\end{pmatrix} + \lambda\left(\begin{pmatrix}2\\4\end{pmatrix} - \begin{pmatrix}-6\\3\end{pmatrix}\right)\\ &= \begin{pmatrix}-6\\3\end{pmatrix} + \lambda\begin{pmatrix}8\\1\end{pmatrix}\\ &= \begin{pmatrix}-6+8\lambda\\3+\lambda\end{pmatrix}\end{aligned}$$

10.
$$\begin{aligned}\mathbf{r} &= \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a})\\ &= \begin{pmatrix}1\\-3\end{pmatrix} + \lambda\left(\begin{pmatrix}-3\\1\end{pmatrix} - \begin{pmatrix}1\\-3\end{pmatrix}\right)\\ &= \begin{pmatrix}1\\-3\end{pmatrix} + \lambda\begin{pmatrix}-4\\4\end{pmatrix}\\ &= \begin{pmatrix}1-4\lambda\\-3+4\lambda\end{pmatrix}\end{aligned}$$

11.
$$\begin{aligned}\mathbf{r} &= \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a})\\ &= \begin{pmatrix}1\\4\end{pmatrix} + \lambda\left(\begin{pmatrix}3\\-1\end{pmatrix} - \begin{pmatrix}1\\4\end{pmatrix}\right)\\ &= \begin{pmatrix}1\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\-5\end{pmatrix}\\ &= \begin{pmatrix}1+2\lambda\\4-5\lambda\end{pmatrix}\end{aligned}$$

12.
$$\begin{aligned}\mathbf{r} &= \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a})\\ &= \begin{pmatrix}5\\0\end{pmatrix} + \lambda\left(\begin{pmatrix}-1\\-4\end{pmatrix} - \begin{pmatrix}5\\0\end{pmatrix}\right)\\ &= \begin{pmatrix}5\\0\end{pmatrix} + \lambda\begin{pmatrix}-6\\-4\end{pmatrix}\\ &= \begin{pmatrix}5-6\lambda\\-4\lambda\end{pmatrix}\end{aligned}$$

13. (a)
$$\begin{aligned}\overrightarrow{\mathrm{AB}} &= (2\mathbf{i} - 3\mathbf{j} + 1(\mathbf{i} - 4\mathbf{j}))\\ &\quad - (2\mathbf{i} - 3\mathbf{j} + -1(\mathbf{i} - 4\mathbf{j}))\\ &= (1 - -1)(\mathbf{i} - 4\mathbf{j})\\ &= 2(\mathbf{i} - 4\mathbf{j})\\ &= 2\mathbf{i} - 8\mathbf{j}\end{aligned}$$

(b)
$$\begin{aligned}|\overrightarrow{\mathrm{BC}}| &= |(2-1)(\mathbf{i} - 4\mathbf{j})|\\ &= |(\mathbf{i} - 4\mathbf{j})|\\ &= \sqrt{1^2 + 4^2}\\ &= \sqrt{17}\end{aligned}$$

(c)
$$\begin{aligned}\overrightarrow{\mathrm{AB}} : \overrightarrow{\mathrm{BC}} &= 2(\mathbf{i} - 4\mathbf{j}) : 1(\mathbf{i} - 4\mathbf{j})\\ &= 2 : 1\end{aligned}$$

14. (a) Vector equation:
$$\begin{aligned}\mathbf{r} &= 5\mathbf{i} - \mathbf{j} + \lambda(7\mathbf{i} + 2\mathbf{j})\\ &= (5+7\lambda)\mathbf{i} + (-1+2\lambda)\mathbf{j}\end{aligned}$$

(b) Parametric equation:
$$x\mathbf{i} + y\mathbf{j} = (5+7\lambda)\mathbf{i} + (-1+2\lambda)\mathbf{j}$$

giving parametric equations
$$\begin{aligned}x &= 5 + 7\lambda\\ y &= -1 + 2\lambda\end{aligned}$$

(c) Cartesian equation:
$$\begin{aligned}2x &= 10 + 14\lambda\\ -7y &= 7 - 14\lambda\\ 2x - 7y &= 17\end{aligned}$$

15. (a) Vector equation:
$$\begin{aligned}\mathbf{r} &= \begin{pmatrix}2\\-1\end{pmatrix} + \lambda\begin{pmatrix}-3\\4\end{pmatrix}\\ &= \begin{pmatrix}2-3\lambda\\-1+4\lambda\end{pmatrix}\end{aligned}$$

(b) Parametric equation:
$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2-3\lambda\\-1+4\lambda\end{pmatrix}$$

giving parametric equations
$$\begin{aligned}x &= 2 - 3\lambda\\ y &= -1 + 4\lambda\end{aligned}$$

(c) Cartesian equation:
$$\begin{aligned}4x &= 8 - 12\lambda\\ 3y &= -3 + 12\lambda\\ 4x + 3y &= 5\end{aligned}$$

16. (a) Vector equation:
$$\begin{aligned}\mathbf{r} &= \begin{pmatrix}0\\3\end{pmatrix} + \lambda\begin{pmatrix}7\\-8\end{pmatrix}\\ &= \begin{pmatrix}7\lambda\\3-8\lambda\end{pmatrix}\end{aligned}$$

(b) Parametric equation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7\lambda \\ 3-8\lambda \end{pmatrix}$$

giving parametric equations

$$\begin{aligned} x &= 7\lambda \\ y &= 3-8\lambda \end{aligned}$$

(c) Cartesian equation:

$$\begin{aligned} 8x &= 56\lambda \\ 7y &= 21-56\lambda \\ 8x+7y &= 21 \end{aligned}$$

17. (a) Vector equation:

$$\begin{aligned} \mathbf{r} &= \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} 2-3\lambda \\ -5+2\lambda \end{pmatrix} \end{aligned}$$

(b) Cartesian equation:

$$\begin{aligned} 2x &= 4-6\lambda \\ 3y &= -15+6\lambda \\ 2x+3y &= -11 \end{aligned}$$

18. (a)

$$\begin{aligned} \overrightarrow{\mathrm{EF}} &= \left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right) \\ &\quad - \left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right) \\ &= (3-2)\begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 3 \end{pmatrix} \end{aligned}$$

(b)

$$\begin{aligned} \overrightarrow{\mathrm{ED}} &= \left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} + -1\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right) \\ &\quad - \left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right) \\ &= (-1-2)\begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ &= -3\begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -9 \end{pmatrix} \end{aligned}$$

(c)

$$\begin{aligned} |\overrightarrow{\mathrm{DE}}| &= |\overrightarrow{\mathrm{ED}}| \\ &= \left|\begin{pmatrix} 3 \\ -9 \end{pmatrix}\right| \\ &= \sqrt{(3)^2 + -9^2} \\ &= \sqrt{90} \\ &= 3\sqrt{10} \end{aligned}$$

(d)

$$\begin{aligned} \overrightarrow{\mathrm{DE}} : \overrightarrow{\mathrm{EF}} &= -\overrightarrow{\mathrm{ED}} : \overrightarrow{\mathrm{EF}} \\ &= -\left(-3\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right) : \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ &= 3\begin{pmatrix} -1 \\ 3 \end{pmatrix} : \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ &= 3 : 1 \end{aligned}$$

(e)

$$\begin{aligned} \overrightarrow{\mathrm{DE}} : \overrightarrow{\mathrm{FE}} &= \overrightarrow{\mathrm{DE}} : -\overrightarrow{\mathrm{EF}} \\ &= 3 : -1 \end{aligned}$$

(f)

$$\begin{aligned} |\overrightarrow{\mathrm{DE}}| : |\overrightarrow{\mathrm{FE}}| &= \left|3\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right| : \left|-\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right| \\ &= 3 : 1 \end{aligned}$$

(Remember, the magnitude of a vector can not be negative.)

19. The vector equation is

$$\mathbf{r} = 7\mathbf{i} - 2\mathbf{j} + \lambda(-2\mathbf{i} + 6\mathbf{j})$$

or

$$\mathbf{r} = (7-2\lambda)\mathbf{i} + (-2+6\lambda)\mathbf{j}$$

Point B: solve for λ for $\mathbf{i}$ and $\mathbf{j}$ components separately.

$$\begin{aligned} 7-2\lambda &= 1 & -2+6\lambda &= 16 \\ \lambda &= 3 & \lambda &= 3 \end{aligned}$$

Point B is on the line.

Point C:

$$\begin{aligned} 7-2\lambda &= 2 & -2+6\lambda &= 13 \\ \lambda &= \frac{5}{2} & \lambda &= \frac{5}{2} \end{aligned}$$

Point C is on the line.

Point D:

$$\begin{aligned} 7-2\lambda &= 8 & -2+6\lambda &= -7 \\ \lambda &= -\frac{1}{2} & \lambda &= -\frac{5}{6} \end{aligned}$$

Point D is not on the line.

Point E:

$$\begin{aligned} 7-2\lambda &= -2 & -2+6\lambda &= 5 \\ \lambda &= \frac{9}{2} & \lambda &= \frac{7}{6} \end{aligned}$$

Point E is not on the line.

20. The vector equation is

$$\mathbf{r} = \begin{pmatrix} 4 \\ -9 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

or

$$\mathbf{r} = \begin{pmatrix} 4-\lambda \\ -9+2\lambda \end{pmatrix}$$

Point G: solve for λ for $\mathbf{i}$ and $\mathbf{j}$ components separately.

$$\begin{aligned} 4-\lambda &= 5 & -9+2\lambda &= 9 \\ \lambda &= -1 & \lambda &= 9 \end{aligned}$$

Point G is not on the line.

Point H:

$$4 - \lambda = 0 \qquad -9 + 2\lambda = -1$$
$$\lambda = 4 \qquad \lambda = 4$$

Point H is on the line.

Point I:

$$4 - \lambda = -3 \qquad -9 + 2\lambda = 5$$
$$\lambda = 7 \qquad \lambda = 7$$

Point I is on the line.

21. (a)
$$\begin{aligned} 3 + 6\lambda &= -3 \\ \lambda &= -1 \\ -1 + 8\lambda &= a \\ a &= -9 \end{aligned}$$

(b)
$$\begin{aligned} -1 + 8\lambda &= 23 \\ \lambda &= 3 \\ 3 + 6\lambda &= b \\ b &= 21 \end{aligned}$$

(c)
$$\begin{aligned} 3 + 6\lambda &= -9 \\ \lambda &= -2 \\ -1 + 8\lambda &= c \\ c &= -17 \end{aligned}$$

(d)
$$\begin{aligned} -1 + 8\lambda &= -21 \\ \lambda &= -\frac{5}{2} \\ 3 + 6\lambda &= d \\ d &= -12 \end{aligned}$$

(e)
$$\begin{aligned} 3 + 6\lambda &= 12 \\ \lambda &= \frac{3}{2} \\ -1 + 8\lambda &= e \\ e &= 11 \end{aligned}$$

(f)
$$\begin{aligned} 3 + 6\lambda &= f \\ -1 + 8\lambda &= f \\ 12 + 24\lambda &= 4f \\ 3 - 24\lambda &= -3f \\ f &= 15 \end{aligned}$$

22. Simply retain the same coefficients for λ:

$$\mathbf{r} = (5 + \lambda)\mathbf{i} + (-6 - \lambda)\mathbf{j}$$

23. Simply retain the same coefficients for λ:

$$\mathbf{r} = \begin{pmatrix} 6 + 3\lambda \\ 5 - 4\lambda \end{pmatrix}$$

24. The parametric equations are:

$$\begin{aligned} x &= 2 + 6t \\ y &= 12 - 10t \end{aligned}$$

Eliminating t:

$$\begin{aligned} 5x &= 10 + 30t \\ 3y &= 36 - 30t \\ 5x + 3y &= 46 \end{aligned}$$

25. At A, on the x-axis, the $\mathbf{j}$ component is zero:

$$\begin{aligned} 8 - 2\lambda &= 0 \\ \lambda &= 4 \\ \text{A} &= 2\mathbf{i} + 8\mathbf{j} + 4(\mathbf{i} - 2\mathbf{j}) \\ &= 6\mathbf{i} \end{aligned}$$

At B, on the y-axis, the $\mathbf{i}$ component is zero:

$$\begin{aligned} 2 + \lambda &= 0 \\ \lambda &= -2 \\ \text{B} &= 2\mathbf{i} + 8\mathbf{j} - 2(\mathbf{i} - 2\mathbf{j}) \\ &= 12\mathbf{j} \end{aligned}$$

26. At A, on the x-axis, the $\mathbf{j}$ component is zero:

$$\begin{aligned} -4 - \lambda &= 0 \\ \lambda &= -4 \\ \text{A} &= \begin{pmatrix} 5 \\ -4 \end{pmatrix} - 4\begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \end{aligned}$$

At B,

$$\begin{aligned} 5 + 2\lambda &= 11 \\ \lambda &= 3 \\ -4 - \lambda &= cc \qquad = -7 \end{aligned}$$

27. Equate L_1 and L_2; solve for λ or μ then substitute back into the corresponding equation to find the point of intersection:

$$\begin{aligned} 14\mathbf{i} - \mathbf{j} + \lambda(5\mathbf{i} - 4\mathbf{i}) &= 9\mathbf{i} - 4\mathbf{j} + \mu(-4\mathbf{i} + 6\mathbf{j}) \\ (14 + 5\lambda - 9 + 4\mu)\mathbf{i} &= (-4 + 6\mu + 1 + 4\lambda)\mathbf{j} \\ (5 + 5\lambda + 4\mu)\mathbf{i} &= (-3 + 4\lambda + 6\mu)\mathbf{j} \\ 5 + 5\lambda + 4\mu &= 0 \\ -3 + 4\lambda + 6\mu &= 0 \\ 15 + 15\lambda + 12\mu &= 0 \\ 6 - 8\lambda - 12\mu &= 0 \\ 21 + 7\lambda &= 0 \\ \lambda &= -3 \\ \mathbf{r} &= 14\mathbf{i} - \mathbf{j} - 3(5\mathbf{i} - 4\mathbf{j}) \\ &= -\mathbf{i} + 11\mathbf{j} \end{aligned}$$

28. Equate L_1 and L_2; solve for λ or μ then substitute back into the corresponding equation to find

the point of intersection:

$$\begin{aligned}\begin{pmatrix}-3\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\end{pmatrix}&=\begin{pmatrix}-10\\2\end{pmatrix}+\mu\begin{pmatrix}-4\\1\end{pmatrix}\\(-3+\lambda+10+4\mu)\mathbf{i}&=(2+\mu-4+\lambda)\mathbf{j}\\(7+\lambda+4\mu)\mathbf{i}&=(-2+\lambda+\mu)\mathbf{j}\\7+\lambda+4\mu&=0\\-2+\lambda+\mu&=0\\9+3\mu&=0\\\mu&=-3\\\mathbf{r}&=\begin{pmatrix}-10\\2\end{pmatrix}-3\begin{pmatrix}-4\\1\end{pmatrix}\\&=\begin{pmatrix}2\\-1\end{pmatrix}\end{aligned}$$

29. Equate L_1 and L_2; solve for λ or μ then substitute back into the corresponding equation to find the point of intersection:

$$\begin{aligned}\begin{pmatrix}-1\\0\end{pmatrix}+\lambda\begin{pmatrix}4\\-10\end{pmatrix}&=\begin{pmatrix}-5\\-9\end{pmatrix}+\mu\begin{pmatrix}1\\7\end{pmatrix}\\(-1+4\lambda+5-\mu)\mathbf{i}&=(-9+7\mu+10\lambda)\mathbf{j}\\(4+4\lambda-\mu)\mathbf{i}&=(-9+10\lambda+7\mu)\mathbf{j}\\4+4\lambda-\mu&=0\\-9+10\lambda+7\mu&=0\\28+28\lambda-7\mu&=0\\19+38\lambda&=0\\\lambda&=-\frac{1}{2}\\\mathbf{r}&=\begin{pmatrix}-1\\0\end{pmatrix}-\frac{1}{2}\begin{pmatrix}4\\-10\end{pmatrix}\\&=\begin{pmatrix}-3\\5\end{pmatrix}\end{aligned}$$

30. Use points A and C to find the vector equation of the line:

$$\begin{aligned}\mathbf{r}&=2\mathbf{i}+3\mathbf{j}+\lambda\left((5\mathbf{i}-4\mathbf{j})-(2\mathbf{i}+3\mathbf{j})\right)\\&=2\mathbf{i}+3\mathbf{j}+\lambda(3\mathbf{i}-7\mathbf{j})\\&=(2+3\lambda)\mathbf{i}+(3-7\lambda)\mathbf{j}\end{aligned}$$

At point B:

$$\begin{aligned}3-7\lambda&=7\\\lambda&=-\frac{4}{7}\\2+3\lambda&=b\\b&=\frac{2}{7}\end{aligned}$$

At point D:

$$\begin{aligned}2+3\lambda&=-2\\\lambda&=-\frac{4}{3}\\3-7\lambda&=d\\d&=\frac{37}{3}\end{aligned}$$

31. $\lambda\begin{pmatrix}1\\4\end{pmatrix}$ and $\mu\begin{pmatrix}2\\c\end{pmatrix}$ both represent the direction of the line, so

$$\lambda\begin{pmatrix}1\\4\end{pmatrix}=\mu\begin{pmatrix}2\\c\end{pmatrix}$$

from the **i** components:

$$\lambda=2\mu$$

so for the **j** components:

$$\begin{aligned}4\lambda&=c\mu\\4(2\mu)&=c\mu\\8\mu&=c\mu\\c&=8\end{aligned}$$

$\begin{pmatrix}9\\d\end{pmatrix}$ is a point on the line $\mathbf{r}=\begin{pmatrix}5\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\4\end{pmatrix}$ so

$$\begin{pmatrix}9\\d\end{pmatrix}=\begin{pmatrix}5\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\4\end{pmatrix}$$

i components:

$$\begin{aligned}9&=5+\lambda\\\lambda&=4\end{aligned}$$

j components:

$$\begin{aligned}d&=3+4\lambda\\&=19\end{aligned}$$

32. $e\mathbf{i}+5\mathbf{j}$ is a point on the line $\mathbf{r}=\mathbf{i}-3\mathbf{j}+\lambda(3\mathbf{i}+4\mathbf{j})$ so

$$e\mathbf{i}+5\mathbf{j}=\mathbf{i}-3\mathbf{j}+\lambda(3\mathbf{i}+4\mathbf{j})$$

j components:

$$\begin{aligned}5&=-3+4\lambda\\\lambda&=2\end{aligned}$$

i components:

$$\begin{aligned}e&=1+3\lambda\\&=7\end{aligned}$$

$\lambda(3\mathbf{i}+4\mathbf{i})$ and $\mu(\mathbf{i}+f\mathbf{j})$ both represent the direction of the line, so

$$\lambda(3\mathbf{i}+4\mathbf{i})=\mu(\mathbf{i}+f\mathbf{j})$$

from the **i** components:

$$3\lambda=\mu$$

so for the **j** components:

$$\begin{aligned}4\lambda&=f\mu\\4\lambda&=f(3\lambda)\\f&=\frac{4}{3}\end{aligned}$$

33. Convert each to a Cartesian equation.

Set ①:

$$\begin{aligned} x &= 1 + 2\lambda \\ y &= \lambda + 3 \\ -2y &= -6 - 2\lambda \\ x - 2y &= -5 \end{aligned}$$

Set ②:

$$\begin{aligned} x &= 2\lambda - 2 \\ y &= 1 + \lambda \\ -2y &= -2\lambda - 2 \\ x - 2y &= -4 \end{aligned}$$

Set ③:

$$\begin{aligned} x &= 8 + 2\lambda \\ y &= 6 + \lambda \\ -2y &= -12 - 2\lambda \\ x - 2y &= -4 \end{aligned}$$

Sets ② and ③ represent the same line. Set ① is the odd one out.

Exercise 4C

1. (a)
$$\begin{aligned} \overrightarrow{AB} &= \mathbf{r}_B - \mathbf{r}_A \\ &= 7\mathbf{i} - 15\mathbf{j} - (-9\mathbf{i} + 5\mathbf{j}) \\ &= (16\mathbf{i} - 20\mathbf{j})\text{m} \end{aligned}$$
(b)
$$\begin{aligned} {}_A\mathbf{v}_B &= \mathbf{v}_A - \mathbf{v}_B \\ &= 5\mathbf{i} - 2\mathbf{j} - (\mathbf{i} + 3\mathbf{j}) \\ &= (4\mathbf{i} - 5\mathbf{j})\text{m/s} \end{aligned}$$
(c) $\overrightarrow{AB} = 4\,{}_A\mathbf{v}_B$
A and B collide at $t = 4$ seconds.

2. (a)
$$\begin{aligned} \overrightarrow{AB} &= \mathbf{r}_B - \mathbf{r}_A \\ &= 11\mathbf{i} - 10\mathbf{j} - (-9\mathbf{i}) \\ &= (20\mathbf{i} - 10\mathbf{j})\text{m} \end{aligned}$$
(b)
$$\begin{aligned} {}_A\mathbf{v}_B &= \mathbf{v}_A - \mathbf{v}_B \\ &= 6\mathbf{i} + 4\mathbf{j} - (-2\mathbf{i} + 8\mathbf{j}) \\ &= (8\mathbf{i} - 4\mathbf{j})\text{m/s} \end{aligned}$$
(c) $\overrightarrow{AB} = 2.5\,{}_A\mathbf{v}_B$
A and B collide at $t = 2.5$ seconds.

3. (a)
$$\begin{aligned} \overrightarrow{AB} &= \mathbf{r}_B - \mathbf{r}_A \\ &= 35\mathbf{i} - 18\mathbf{j} - (-\mathbf{i} + 30\mathbf{j}) \\ &= (36\mathbf{i} - 48\mathbf{j})\text{m} \end{aligned}$$
(b)
$$\begin{aligned} {}_A\mathbf{v}_B &= \mathbf{v}_A - \mathbf{v}_B \\ &= 9\mathbf{i} + 2\mathbf{j} - (3\mathbf{i} + 10\mathbf{j}) \\ &= (6\mathbf{i} - 8\mathbf{j})\text{m/s} \end{aligned}$$
(c) $\overrightarrow{AB} = 6\,{}_A\mathbf{v}_B$
A and B collide at $t = 6$ seconds.

4. (a)
$$\begin{aligned} \overrightarrow{AB} &= \mathbf{r}_B - \mathbf{r}_A \\ &= 56\mathbf{i} + 4\mathbf{j} - (10\mathbf{i} + 32\mathbf{j}) \\ &= (46\mathbf{i} - 28\mathbf{j})\text{m} \end{aligned}$$
(b)
$$\begin{aligned} {}_A\mathbf{v}_B &= \mathbf{v}_A - \mathbf{v}_B \\ &= 10\mathbf{i} + 2\mathbf{j} - (-2\mathbf{i} + 12\mathbf{j}) \\ &= (12\mathbf{i} - 10\mathbf{j})\text{m/s} \end{aligned}$$
(c) $\overrightarrow{AB}$ is not a scalar multiple of ${}_A\mathbf{v}_B$ since solving $\overrightarrow{AB} = t\,{}_A\mathbf{v}_B$ for t gives $t = \frac{23}{6}$ for the $\mathbf{i}$ components and $t = \frac{14}{5}$ for the $\mathbf{j}$ components.
A and B do not collide.

5. (a)
$$\begin{aligned} \overrightarrow{AB} &= \mathbf{r}_B - \mathbf{r}_A \\ &= 30\mathbf{i} - 10\mathbf{j} - (-2\mathbf{i} + 22\mathbf{j}) \\ &= (32\mathbf{i} - 32\mathbf{j})\text{m} \end{aligned}$$
(b)
$$\begin{aligned} {}_A\mathbf{v}_B &= \mathbf{v}_A - \mathbf{v}_B \\ &= 4\mathbf{i} + 6\mathbf{j} - (12\mathbf{i} - 2\mathbf{j}) \\ &= (-8\mathbf{i} + 8\mathbf{j})\text{m/s} \end{aligned}$$
(c) $\overrightarrow{AB}$ is a scalar multiple of ${}_A\mathbf{v}_B$ but solving $\overrightarrow{AB} = t\,{}_A\mathbf{v}_B$ for t gives $t = -4$. This suggests that rather than heading for a collision A and B are diverging from a common point from which they originated 4 seconds before time $t = 0$.
A and B do not collide.

6. (a)
$$\begin{aligned} \overrightarrow{PQ} &= \mathbf{r}_Q - \mathbf{r}_P \\ &= 10\mathbf{i} + 20\mathbf{j} - (25\mathbf{i} - 22\mathbf{j}) \\ &= (-15\mathbf{i} + 42\mathbf{j})\text{m} \end{aligned}$$
(b) $\overrightarrow{PQ}$ is the displacement of Q relative to P (i.e. from P to Q).

(c)
$$\begin{aligned} {}_{A}\mathbf{v}_{B} &= \mathbf{v}_{A} - \mathbf{v}_{B} \\ &= 20\mathbf{i} + 44\mathbf{j} - (40\mathbf{i} - 12\mathbf{j}) \\ &= (-20\mathbf{i} + 56\mathbf{j})\text{m/s} \\ \overrightarrow{PQ} &= t\,{}_{A}\mathbf{v}_{B} \\ -15\mathbf{i} + 42\mathbf{j} &= t(-20\mathbf{i} + 56\mathbf{j}) \end{aligned}$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components separately:

$$-15 = -20t \qquad 42 = 56t$$
$$t = \frac{3}{4} \qquad t = \frac{3}{4}$$

A and B will collide at $t = 0.75$s.

7. (a) The position of A at 8:00am is

$$\begin{aligned} \mathbf{r}_A &= 5\mathbf{i} + 28\mathbf{j} + 0.5(20\mathbf{i} + 4\mathbf{j}) \\ &= (15\mathbf{i} + 30\mathbf{j})\text{km} \end{aligned}$$

$$\begin{aligned} \overrightarrow{BA} &= \mathbf{r}_A - \mathbf{r}_B \\ &= (15\mathbf{i} + 30\mathbf{j}) - (20\mathbf{i} + 70\mathbf{j}) \\ &= (-5\mathbf{i} - 40\mathbf{j})\text{km} \end{aligned}$$

(b)
$$\begin{aligned} {}_{B}\mathbf{v}_{A} &= \mathbf{v}_{B} - \mathbf{v}_{A} \\ &= (x\mathbf{i} - 12\mathbf{j}) - (20\mathbf{i} + 4\mathbf{j}) \\ &= ((x - 20)\mathbf{i} - 16\mathbf{j})\text{km/h} \\ \overrightarrow{BA} &= t\,{}_{B}\mathbf{v}_{A} \\ -5\mathbf{i} - 40\mathbf{j} &= t((x - 20)\mathbf{i} - 16\mathbf{j}) \end{aligned}$$

Equating $\mathbf{j}$ components:

$$\begin{aligned} -40 &= -16t \\ t &= 2.5\,\text{hours} \end{aligned}$$

A and B will collide 2.5 hours after 8:00am, i.e. at 10:30am.
Equating $\mathbf{i}$ components:

$$\begin{aligned} -5 &= t(x - 20) \\ -5 &= 2.5(x - 20) \\ x - 20 &= -2 \\ x &= 18 \end{aligned}$$

8. Let $\mathbf{j}$ represent 1km due north and $\mathbf{i}$ 1km due east.

$$\begin{aligned} \overrightarrow{BA} &= 6\mathbf{j}\text{km} \\ \mathbf{v}_A &= (p\mathbf{i})\text{km/h} \\ \mathbf{v}_B &= (8\sqrt{2}\cos 45°\mathbf{i} + 8\sqrt{2}\sin 45°\mathbf{j}) \\ &= (8\mathbf{i} + 8\mathbf{j})\text{km/h} \\ {}_{B}\mathbf{v}_{A} &= \mathbf{v}_B - \mathbf{v}_A \\ &= (8\mathbf{i} + 8\mathbf{j}) - (p\mathbf{i}) \\ &= ((8 - p)\mathbf{i} + 8\mathbf{j})\text{km/h} \\ \overrightarrow{BA} &= t\,{}_{B}\mathbf{v}_{A} \\ 6\mathbf{j} &= t((8 - p)\mathbf{i} + 8\mathbf{j}) \end{aligned}$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components separately:

$$0 = t(8 - p) \qquad 6 = 8t$$
$$p = 8 \qquad t = 0.75$$

Tankers collide in $\frac{3}{4}$ hour, at 8:45am.

9. Let F represent the fishing vessel and C represent the coastguard boat. Let the desired velocity of the coastguard boat be $(a\mathbf{i} + b\mathbf{j})$km/h.

$$\begin{aligned} \overrightarrow{CF} &= (21\mathbf{i} + 21\mathbf{j})\text{km} \\ {}_{C}\mathbf{v}_{F} &= \mathbf{v}_C - \mathbf{v}_F \\ &= (a\mathbf{i} + b\mathbf{j}) - (8\mathbf{i} - 4\mathbf{j}) \\ &= ((a - 8)\mathbf{i} + (b + 4)\mathbf{j})\text{km/h} \end{aligned}$$

$$\begin{aligned} \overrightarrow{CF} &= t\,{}_{C}\mathbf{v}_{F} \\ 21\mathbf{i} + 21\mathbf{j} &= t((a - 8)\mathbf{i} + (b + 4)\mathbf{j}) \\ 21 &= t(a - 8) \\ 21 &= t(b + 4) \\ a - 8 &= b + 4 \\ a &= b + 12 \end{aligned}$$

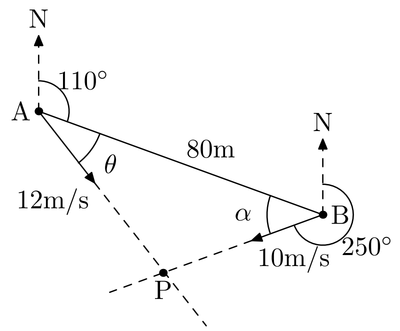
$$\begin{aligned} a^2 + b^2 &= (4\sqrt{29})^2 \\ (b + 12)^2 + b^2 &= 464 \\ 2b^2 + 24b - 320 &= 0 \\ 2(b - 8)(b + 20) &= 0 \\ b &= 8 \end{aligned}$$

(disregarding the root at $b = -20$ because this would result in a negative value of time when substituted into $21 = t(b - 4)$)

$$\begin{aligned} a &= b + 12 \\ &= 20 \\ 21 &= t(b + 4) \\ 12t &= 21 \\ t &= \frac{7}{4} \end{aligned}$$

The patrol boat should set a velocity of $(20\mathbf{i} + 8\mathbf{j})$km/h in order to intercept the fishing vessel in $1\frac{3}{4}$ hours, i.e. at 1:45pm.

10. Let P be the point where the projectiles collide.



$$\begin{aligned}\alpha &= 360 - 250 - (180 - 110)\\ &= 40^\circ\\ \text{BP} &= 10t\\ \text{AP} &= 12t\\ \frac{\sin\theta}{10t} &= \frac{\sin\alpha}{12t}\\ \sin\theta &= \frac{10t\sin 40^\circ}{12t}\\ \theta &= \sin^{-1}\frac{5\sin 40^\circ}{6}\\ &= 32.4^\circ\end{aligned}$$

The second object should be projected on a bearing of $110 + 32 = 142^\circ$.

$$\begin{aligned}\frac{\text{AP}}{\sin\alpha} &= \frac{80}{\sin(180-\alpha-\theta)}\\ \text{AP} &= \frac{80\sin\alpha}{\sin(180-\alpha-\theta)}\\ &= \frac{80\sin 40^\circ}{\sin(180-40-32.4)^\circ}\\ &= 54.0\text{m}\\ t &= \frac{54}{12}\\ &= 4.5\text{s}\end{aligned}$$

You could also do this using component vectors, but it's rather more work:

Let $\mathbf{i}$ represent 1m due east and $\mathbf{j}$ represent 1m due north.

$$\begin{aligned}\overrightarrow{\text{AB}} &= 80\sin 110^\circ + 80\cos 110^\circ\\ &= 75.1754\mathbf{i} - 27.3616\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{v}_\text{B} &= 10\sin 250^\circ\mathbf{i} + 10\cos 250^\circ\\ &= -9.3969\mathbf{i} - 3.4202\mathbf{j}\end{aligned}$$

$$\mathbf{v}_\text{A} = a\mathbf{i} + b\mathbf{j}$$

$$\begin{aligned}{}_\text{A}\mathbf{v}_\text{B} &= \mathbf{v}_\text{A} - \mathbf{v}_\text{B}\\ &= (a\mathbf{i}+b\mathbf{j}) - (-9.3969\mathbf{i} - 3.4202\mathbf{j})\\ &= (a+9.3969)\mathbf{i} + (b+3.4202)\mathbf{j}\end{aligned}$$

$$\begin{aligned}\overrightarrow{\text{AB}} &= t_\text{A}\mathbf{v}_\text{B}\\ 75.1754\mathbf{i} - 27.3616\mathbf{j} &= t((a+9.3969)\mathbf{i}\\ &\quad + (b+3.4202)\mathbf{j})\\ 75.1754 &= t(a+9.3969)\\ t &= \frac{75.1754}{a+9.3969}\\ -27.3616 &= t(b+3.4202)\\ t &= \frac{-27.3616}{b+3.4202}\\ \frac{75.1754}{a+9.3969} &= \frac{-27.3616}{b+3.4202}\\ 75.1754(b+3.4202) &= -27.3616(a+9.3969)\\ b+3.4202 &= -0.3640(a+9.3969)\\ &= -0.3640a - 3.4202\\ b &= -0.3640a - 6.8404\end{aligned}$$

$$\begin{aligned}a^2+b^2 &= 12^2\\ a^2 + (-0.3640a - 6.8404)^2 &= 144\\ a &= -11.7206\\ \text{or } a &= 7.3237\end{aligned}$$

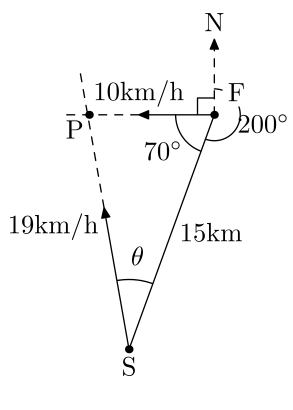
The first root will give negative t when substituted into $t = \frac{75.1754}{a+9.3969}$ so we can discard it.

$$\begin{aligned}t &= \frac{75.1754}{(7.3237)+9.3969}\\ &= 4.4960\\ b &= -0.3640(7.3237) - 6.8404\\ &= -9.5060\\ \mathbf{v}_\text{A} &= 7.3237\mathbf{i} - 9.5060\mathbf{j}\\ \tan\theta &= \frac{7.3237}{-9.5060}\\ \theta &= 142.39^\circ\end{aligned}$$

(Although $\tan^{-1}\frac{7.3237}{-9.5060}$ on the calculator gives -37.61° that is an angle with negative sine and positive cosine; we want the angle with positive sine and negative cosine, so we must add 180°.)

The object should be projected on a bearing of 142° and collision will occur after 4.5s.

11. Let F represent the *Big Freezer* and S the *Jolly Snapper*. Let P be the point of interception.



$$\begin{aligned}
\text{FP} &= 10t \\
\text{SP} &= 19t \\
\frac{\sin\theta}{10t} &= \frac{\sin 70^\circ}{19t} \\
\sin\theta &= \frac{10t\sin 70^\circ}{19t} \\
\theta &= \sin^{-1}\frac{10\sin 70^\circ}{19} \\
&= 29.6^\circ
\end{aligned}$$

The Jolly Snapper should steam on a bearing of $(200+180)-30 = 350^\circ$.

$$\begin{aligned}
\frac{\text{SP}}{\sin 70^\circ} &= \frac{15}{\sin(180-70-\theta)} \\
\text{SP} &= \frac{15\sin 70^\circ}{\sin(80.4^\circ)} \\
&= 14.3\text{km} \\
t &= \frac{14.3}{19} \\
&= 0.752\text{hours} \\
&\approx 45\text{minutes}
\end{aligned}$$

Jolly Snapper should be travel on a bearing of 350° and will intercept *Big Freezer* after 0.752 hours, that is at about 6:45am.

Again with component vectors:

Let $\mathbf{i}$ represent 1km due east and $\mathbf{j}$ represent 1km due north.

$$\begin{aligned}
\overrightarrow{\text{FS}} &= 15\sin 200^\circ + 15\cos 200^\circ \\
&= -5.130\mathbf{i} - 14.095\mathbf{j}
\end{aligned}$$

$$\mathbf{v}_\text{F} = -10\mathbf{i}$$

$$\mathbf{v}_\text{S} = a\mathbf{i} + b\mathbf{j}$$

$$\begin{aligned}
{}_\text{F}\mathbf{v}_\text{S} &= \mathbf{v}_\text{F} - \mathbf{v}_\text{S} \\
&= -10\mathbf{i} - (a\mathbf{i} + b\mathbf{j}) \\
&= (-10-a)\mathbf{i} - b\mathbf{j}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{\text{FS}} &= t_\text{F}\mathbf{v}_\text{S} \\
-5.130\mathbf{i} - 14.095\mathbf{j} &= t((-10-a)\mathbf{i} - b\mathbf{j}) \\
-5.130 &= t(-10-a) \\
t &= \frac{-5.130}{-10-a} \\
-14.095 &= -tb \\
t &= \frac{-14.095}{-b} \\
\frac{-5.130}{-10-a} &= \frac{-14.095}{-b} \\
5.130b &= 14.095(10+a) \\
b &= 2.7475(10+a)
\end{aligned}$$

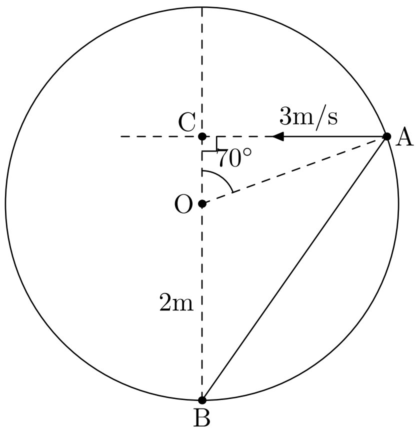
$$\begin{aligned}
a^2 + b^2 &= 19^2 \\
a^2 + (2.7475(10+a))^2 &= 19^2 \\
a &= -14.478 \\
\text{or } a &= -3.182
\end{aligned}$$

The first root will give negative t when substituted into $t = \frac{-14.095}{-10-a}$ so we can discard it.

$$\begin{aligned}
t &= \frac{-5.130}{-10-(-3.182)} \\
&= 0.752\text{hours} \\
b &= 2.7475(10+(-3.182)) \\
&= 18.732 \\
\mathbf{v}_\text{S} &= -3.182\mathbf{i} + 18.732\mathbf{j} \\
\tan\theta &= \frac{-3.182}{18.732} \\
\theta &= -9.64^\circ
\end{aligned}$$

Jolly Snapper should be travel on a bearing of 350° and will intercept *Big Freezer* after 0.752 hours, that is at about 6:45am.

12. Let C be the point where the marble from A crosses the diameter from B, thus:



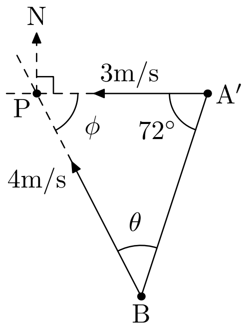
$$\begin{aligned}
\text{AC} &= 2\sin 70^\circ \\
&= 1.88\text{m} \\
\text{BC} &= 2 + 2\cos 70^\circ \\
&= 2.68\text{m}
\end{aligned}$$

Let A′ be the position of the first marble after it has travelled one metre.

$$\begin{aligned}
\text{A}'\text{C} &= \text{AC} - 1 \\
&= 0.88\text{m} \\
\tan\angle\text{BA}'\text{C} &= \frac{\text{BC}}{\text{A}'\text{C}} \\
\angle\text{BA}'\text{C} &= \tan^{-1}\frac{2.68}{0.88} \\
&= 71.9^\circ
\end{aligned}$$

Let P be the point where the collision occurs.

Redrawing with only the salient features gives:



$$\begin{aligned}\frac{\sin\theta}{3t} &= \frac{\sin 71.9^\circ}{4t}\\ \sin\theta &= \frac{3t\sin 71.9^\circ}{4t}\\ \theta &= \sin^{-1}(0.75\sin 71.9^\circ)\\ &= 45.5^\circ\\ \phi &= 180 - 71.9 - 45.5\\ &= 62.7^\circ\\ &\approx 63^\circ\end{aligned}$$

Thus the bearing from P to B is $90 + 63 = 153^\circ$ and the bearing from B to P is $153 + 180 = 333^\circ$.

B should roll her marble on a bearing of 333°.

Using component vectors here again results in similar or greater level of complexity:

The position of child A's marble before it is played is:

$$\begin{aligned}\mathbf{r}_\mathrm{A} &= 2\sin 70^\circ\mathbf{i} + 2\cos 70^\circ\mathbf{j}\\ &= (1.879\mathbf{i} + 0.684\mathbf{j})\mathrm{m}\end{aligned}$$

After it has rolled 1m this becomes

$$\begin{aligned}\mathbf{r}_\mathrm{A} &= 1.879\mathbf{i} + 0.684\mathbf{j} - 1\mathbf{i}\\ &= (0.879\mathbf{i} + 0.684\mathbf{j})\mathrm{m}\end{aligned}$$

Marble A has velocity

$$\mathbf{v}_\mathrm{A} = (-3\mathbf{i})\mathrm{m/s}$$

The position of child B's marble is initially

$$\mathbf{r}_\mathrm{B} = -2\mathbf{j}\mathrm{m}$$

Let $\mathbf{v}_\mathrm{B} = (a\mathbf{i} + b\mathbf{j})\mathrm{m/s}$.

$$\begin{aligned}\overrightarrow{\mathrm{BA}} &= \mathbf{r}_\mathrm{A} - \mathbf{r}_\mathrm{B}\\ &= 0.879\mathbf{i} + 0.684\mathbf{j} - (-2\mathbf{j})\\ &= 0.879\mathbf{i} + 2.684\mathbf{j}\end{aligned}$$

$$\begin{aligned}{}_\mathrm{B}\mathbf{v}_\mathrm{A} &= \mathbf{v}_\mathrm{B} - \mathbf{v}_\mathrm{A}\\ &= (a\mathbf{i} + b\mathbf{j}) - (-3\mathbf{i})\\ &= (a+3)\mathbf{i} + b\mathbf{j}\end{aligned}$$

$$\begin{aligned}\overrightarrow{\mathrm{BA}} &= t_\mathrm{B}\mathbf{v}_\mathrm{A}\\ 0.879\mathbf{i} + 2.684\mathbf{j} &= t((a+3)\mathbf{i} + b\mathbf{j})\\ 0.879 &= t(a+3)\\ t &= \frac{0.879}{a+3}\\ 2.684 &= tb\\ t &= \frac{2.684}{b}\\ \frac{0.879}{a+3} &= \frac{2.684}{b}\\ 0.879b &= 2.684(a+3)\\ b &= 3.052(a+3)\end{aligned}$$

$$\begin{aligned}a^2 + b^2 &= 4^2\\ a^2 + (3.052(a+3))^2 &= 4^2\\ a &= -3.583\\ \text{or } a &= -1.836\end{aligned}$$

The first root will give negative t when substituted into $t = \frac{0.879}{a+3}$ so we can discard it.

(We don't actually need to calculate t since the question does not ask for it, but here it is anyway.)

$$\begin{aligned}t &= \frac{0.879}{(-1.836) + 3}\\ &= 0.755\mathrm{s}\end{aligned}$$

$$\begin{aligned}b &= 3.052((-1.836) + 3)\\ &= 3.554\\ \mathbf{v}_\mathrm{B} &= -1.836\mathbf{i} + 3.554\mathbf{j}\\ \tan\theta &= \frac{-1.836}{3.554}\\ \theta &= -27.32^\circ\end{aligned}$$

Child B should play her marble on a bearing of $360 - 27 = 333^\circ$.

Exercise 4D

1. (a) $|\mathbf{r}| = 13$

   (b) Point A:

$$\begin{aligned}|(7\mathbf{i} - 7\mathbf{j})| &= \sqrt{7^2 + 7^2}\\ &\approx 9.9\\ 9.9 &< 13\end{aligned}$$

   Point A lies inside the circle.
   Point B:

$$\begin{aligned}|(12\mathbf{i} - 5\mathbf{j})| &= \sqrt{12^2 + 5^2}\\ &= 13\end{aligned}$$

   Point B lies on the circle.

2. A: This looks like it might be a circle, but we need to check to make sure it has a positive radius:

$$\begin{aligned}x^2 + y^2 - 2x + 4y &= 6\\ (x-1)^2 - 1 + (y+2)^2 - 4 &= 6\\ (x-1)^2 + (y+2)^2 &= 11\end{aligned}$$

   This represents a circle radius $\sqrt{11}$ centred at $(1, -2)$.

   B: This is not a circle because the x^2 and y^2 terms have different coefficients. (It might be an ellipse.)

   C: This represents a circle radius $\sqrt{6}$ centred at the origin.

   D: This looks like it might be a circle. Checking:

$$\begin{aligned}x^2 + y^2 + 8x &= 10\\ (x-4)^2 - 16 + y^2 &= 10\\ (x-4)^2 + y^2 &= 26\end{aligned}$$

   This represents a circle radius $\sqrt{26}$ centred at $(4, 0)$.

   E: This is not a circle because the x^2 and y^2 terms have different coefficients. (It might be a hyperbola.)

   F: This is not a circle because it has an xy term.

3. (a) $|r| = 25$

   (b) Point A:

$$\begin{aligned}|19\mathbf{i} - 18\mathbf{j}| &= \sqrt{19^2 + 18^2}\\ &\approx 26.2\\ 26.2 &> 25\end{aligned}$$

   Point A lies outside the circle.

   Point B:

$$\begin{aligned}|-20\mathbf{i} + 15\mathbf{j}| &= \sqrt{20^2 + 15^2}\\ &= 25\end{aligned}$$

   Point B lies on the circle.
   Point C:

$$\begin{aligned}|14\mathbf{i} + 17\mathbf{j}| &= \sqrt{14^2 + 17^2}\\ &\approx 22.0\\ 22.0 &< 25\end{aligned}$$

   Point C lies inside the circle.
   Point D:

$$\begin{aligned}|-24\mathbf{i} - 7\mathbf{j}| &= \sqrt{24^2 + 7^2}\\ &= 25\end{aligned}$$

   Point D lies on the circle.

4. $x^2 + y^2 = 100$

   Point A:

$$\begin{aligned}(-6)^2 + a^2 &= 100\\ a &= \sqrt{100 - 36}\\ &= 8\end{aligned}$$

   Point B:

$$\begin{aligned}3^2 + b^2 &= 100\\ b &= \sqrt{100 - 9}\\ &= \sqrt{91}\end{aligned}$$

   Point C:

$$\begin{aligned}0^2 + c^2 &= 100\\ c &= -\sqrt{100 - 0}\\ &= -10\end{aligned}$$

   Point D:

$$\begin{aligned}d^2 + 5^2 &= 100\\ d &= -\sqrt{100 - 25}\\ &= -\sqrt{75}\\ &= -5\sqrt{3}\end{aligned}$$

5. The equation is

$$|\mathbf{r} - (-7\mathbf{i} + 4\mathbf{j})| = 4\sqrt{5}$$

$$\begin{aligned}|(\mathbf{i} + 8\mathbf{j}) - (-7\mathbf{i} + 4\mathbf{j})| &= |8\mathbf{i} + 4\mathbf{j}|\\ &= \sqrt{8^2 + 4^2}\\ &= 4\sqrt{5}\end{aligned}$$

   Point A lies on the circle.

6. (a) $(x-2)^2 + (y - -3)^2 = 5^2$

$$(x-2)^2 + (y+3)^2 = 25$$

(b) $(x-3)^2+(y-2)^2 = 7^2$
$(x-3)^2+(y-2)^2 = 49$

(c) $(x--10)^2+(y-2)^2 = (3\sqrt{5})^2$
$(x+10)^2+(y-2)^2 = 45$

(d) $(x--1)^2+(y--1)^2 = 6^2$
$(x+1)^2+(y+1)^2 = 36$

7. (a)
$$\begin{aligned}(x-3)^2+(y-5)^2 &= 5^2\\ (x^2-6x+9)+(y^2-10y+25)^2 &= 25\\ x^2+y^2-6x-10y+34 &= 25\\ x^2+y^2-6x-10y &= -9\end{aligned}$$

(b)
$$\begin{aligned}(x--2)^2+(y-1)^2 &= (\sqrt{7})^2\\ (x+2)^2+(y-1)^2 &= 7\\ x^2+4x+4+y^2-2y+1 &= 7\\ x^2+y^2+4x-2y+5 &= 7\\ x^2+y^2+4x-2y &= 2\end{aligned}$$

(c)
$$\begin{aligned}(x--3)^2+(y--1)^2 &= 2^2\\ (x+3)^2+(y+1)^2 &= 4\\ x^2+6x+9+y^2+2y+1 &= 4\\ x^2+y^2+6x+2y+10 &= 4\\ x^2+y^2+6x+2y &= -6\end{aligned}$$

(d)
$$\begin{aligned}(x-3)^2+(y-8)^2 &= (2\sqrt{7})^2\\ x^2-6x+9+y^2-16y+64 &= 28\\ x^2+y^2-6x-16y+73 &= 28\\ x^2+y^2-6x-16y &= -45\end{aligned}$$

8. (a) Radius $=\sqrt{5}$; centre $=(6,3)$

(b) $|\mathbf{r}-2\mathbf{i}+3\mathbf{j}| = 6$
$|\mathbf{r}-(2\mathbf{i}-3\mathbf{j})| = 6$
Radius $=6$, centre $=(2,-3)$

(c) Radius $=3$, centre $=(3,-4)$

(d) Radius $=5$, centre $=(0,0)$

(e)
$$\begin{aligned}25x^2+25y^2 &= 9\\ 25(x^2+y^2) &= 9\\ x^2+y^2 &= \frac{9}{25}\end{aligned}$$
Radius $=\frac{3}{5}$, centre $=(0,0)$

(f) Radius $=5$, centre $=(3,-4)$

(g) Radius $=10$, centre $=(-7,1)$

(h) Radius $=20$, centre $=(0,0)$

(i)
$$\begin{aligned}x^2+y^2-6x+4y+4 &= 0\\ (x-3)^2-9+(y+2)^2-4+4 &= 0\\ (x-3)^2+(y+2)^2 &= 9\end{aligned}$$
Radius $=3$, centre $=(3,-2)$

(j)
$$\begin{aligned}x^2+y^2+2x-6y &= 15\\ (x+1)^2-1+(y-3)^2-9 &= 15\\ (x+1)^2+(y-3)^2 &= 25\end{aligned}$$
Radius $=5$, centre $=(-1,3)$

9. The first circle has centre $3\mathbf{i}+7\mathbf{j}$. The second circle has centre $2\mathbf{i}+9\mathbf{j}$. The distance between these centres is
$$\begin{aligned}d &= \sqrt{(3-2)^2+(7-9)^2}\\ &= \sqrt{1^2+(-2)^2}\\ &= \sqrt{5}\end{aligned}$$

10.
$$\begin{aligned}\mathrm{A} &= 3\mathbf{i}-4\mathbf{j}\\ \mathrm{B} &= 2\mathbf{i}+7\mathbf{j}\end{aligned}$$
The line through A and B is given by
$$\begin{aligned}\mathbf{r} &= \mathrm{A}+\lambda\overrightarrow{\mathrm{AB}}\\ &= 3\mathbf{i}-4\mathbf{j}+\lambda(2\mathbf{i}+7\mathbf{j}-(3\mathbf{i}-4\mathbf{j}))\\ &= 3\mathbf{i}-4\mathbf{j}+\lambda(-\mathbf{i}+11\mathbf{j})\\ &= (3-\lambda)\mathbf{i}+(-4+11\lambda)\mathbf{j}\end{aligned}$$

11.
$$\begin{aligned}\mathrm{A} &= 3\mathbf{i}+11\mathbf{j}\\ \mathrm{B} &= 12\mathbf{i}-\mathbf{j}\\ |\overrightarrow{\mathrm{AB}}| &= \sqrt{(3-12)^2+(11--1)^2}\\ &= \sqrt{9^2+12^2}\\ &= 15\end{aligned}$$
The distance between centres is 15. The circle centred at A has a radius of 12 and that centred at B has a radius of 3; 3+12=15: the circles touch at just one point (i.e. one point in common).

12.
$$\begin{aligned}\mathrm{A} &= 2\mathbf{i}+3\mathbf{j}\\ \mathrm{B} &= -2\mathbf{i}+5\mathbf{j}\\ |\overrightarrow{\mathrm{AB}}| &= \sqrt{(2--2)^2+(3-5)^2}\\ &= \sqrt{4^2+2^2}\\ &= \sqrt{20}\end{aligned}$$
The distance between centres is $\sqrt{20}$. The circle centred at A has a radius of 3 and that centred at B has a radius of 1; $3+1=4=\sqrt{16}<\sqrt{20}$: the circles are further apart than the sum of their radii, so they do not intersect (i.e. no points in common).

13. Substitute the expression for $\mathbf{r}$ in the equation of the line into the equation of the circle:
$$\begin{aligned}\left|\begin{pmatrix}6\\9\end{pmatrix}+\lambda\begin{pmatrix}4\\1\end{pmatrix}-\begin{pmatrix}-5\\2\end{pmatrix}\right| &= \sqrt{34}\\ \left|\begin{pmatrix}11+4\lambda\\7+\lambda\end{pmatrix}\right| &= \sqrt{34}\\ (11+4\lambda)^2+(7+\lambda)^2 &= 34\\ 121+88\lambda+16\lambda^2+49+14\lambda+\lambda^2 &= 34\\ 170+102\lambda+17\lambda^2 &= 34\\ 17\lambda^2+102\lambda+136 &= 0\\ \lambda^2+6\lambda+8 &= 0\\ (\lambda+4)(\lambda+2) &= 0\\ \lambda &= -4\\ \text{or } \lambda &= -2\end{aligned}$$

$$\begin{aligned}
r &= \begin{pmatrix} 6 \\ 9 \end{pmatrix} - 4\begin{pmatrix} 4 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 6 \\ 9 \end{pmatrix} - \begin{pmatrix} 16 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} -10 \\ 5 \end{pmatrix} \\
\text{or } r &= \begin{pmatrix} 6 \\ 9 \end{pmatrix} - 2\begin{pmatrix} 4 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 6 \\ 9 \end{pmatrix} - \begin{pmatrix} 8 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} -2 \\ 7 \end{pmatrix}
\end{aligned}$$

14. Substitute the expression for $\mathbf{r}$ in the equation of the line into the equation of the circle:

$$\begin{aligned}
|-\mathbf{i} + 8\mathbf{j} + \lambda(6\mathbf{i} + 2\mathbf{j}) - (7\mathbf{i} + 4\mathbf{j})| &= \sqrt{40} \\
|-8\mathbf{i} + 4\mathbf{j} + \lambda(6\mathbf{i} + 2\mathbf{j})| &= \sqrt{40} \\
|(-8 + 6\lambda)\mathbf{i} + (4 + 2\lambda)\mathbf{j}| &= \sqrt{40} \\
(-8 + 6\lambda)^2 + (4 + 2\lambda)^2 &= 40 \\
64 - 96\lambda + 36\lambda^2 + 16 + 16\lambda + 4\lambda^2 &= 40 \\
40\lambda^2 - 80\lambda + 80 &= 40 \\
40\lambda^2 - 80\lambda + 40 &= 0 \\
\lambda^2 - 2\lambda + 1 &= 0 \\
(\lambda - 1)^2 &= 0 \\
\lambda &= 1
\end{aligned}$$

$$\begin{aligned}
r &= -\mathbf{i} + 8\mathbf{j} + 1(6\mathbf{i} + 2\mathbf{j}) \\
&= 5\mathbf{i} + 10\mathbf{j}
\end{aligned}$$

Miscellaneous Exercise 4

1. Let $z = a + b\text{i}$ where a and b are real.

$$\begin{aligned}
3z + 2\bar{z} &= 5 + 5\text{i} \\
3(a + b\text{i}) + 2(a - b\text{i}) &= 5 + 5\text{i} \\
3a + 3b\text{i} + 2a - 2b\text{i} &= 5 + 5\text{i} \\
5a + b\text{i} &= 5 + 5\text{i} \\
a &= 1 \\
b &= 5 \\
z &= 1 + 5\text{i}
\end{aligned}$$

2. Let $z = a + b\text{i}$ where a and b are real.

$$\begin{aligned}
z(2 - 3\text{i}) &= 5 + \text{i} \\
(a + b\text{i})(2 - 3\text{i}) &= 5 + \text{i} \\
2a - 3a\text{i} + 2b\text{i} + 3b &= 5 + \text{i} \\
2a + 3b + (-3a + 2b)\text{i} &= 5 + \text{i} \\
2a + 3b &= 5 \\
\text{and } -3a + 2b &= 1 \\
6a + 9b &= 15 \\
-6a + 4b &= 2 \\
13b &= 17 \\
b &= \frac{17}{13} \\
-3a + 2\left(\frac{17}{13}\right) &= 1 \\
-39a + 34 &= 13 \\
-39a &= -21 \\
13a &= 7 \\
a &= \frac{7}{13} \\
z &= \frac{7}{13} + \frac{17}{13}\text{i}
\end{aligned}$$

3. Left Hand Side:

$$\begin{aligned}
&2\sin^3\theta\cos\theta + 2\cos^3\theta\sin\theta \\
&\qquad = 2\sin\theta\cos\theta(\sin^2\theta+\cos^2\theta) \\
&\qquad = 2\sin\theta\cos\theta \\
&\qquad = \sin 2\theta \\
&\qquad = \text{R.H.S.}
\end{aligned}$$

□

4.
$$\begin{aligned}
2\sin x\cos x &= \sqrt{3}(1-2\sin^2 x) \\
\sin 2x &= \sqrt{3}\cos 2x \\
\tan 2x &= \sqrt{3} \\
2x &= 60^\circ \\
\text{or } 2x &= 180+60 \\
&= 240^\circ \\
\text{or } 2x &= 360+60 \\
&= 420^\circ \\
\text{or } 2x &= 540+60 \\
&= 600^\circ \\
x &= 30^\circ \\
\text{or } x &= 120^\circ \\
\text{or } x &= 210^\circ \\
\text{or } x &= 300^\circ
\end{aligned}$$

5. Let two complex numbers be $a+b\text{i}$ and $c+d\text{i}$.

   The sum of the conjugates is

$$a - b\text{i} + c - d\text{i} = (a+c) - (b+d)\text{i}$$

   The conjugate of the sum is

$$\begin{aligned}
\overline{a+b\text{i}+c+d\text{i}} &= \overline{(a+c)+(b+d)\text{i}} \\
&= (a+c)-(b+d)\text{i}
\end{aligned}$$

   Hence the sum of the conjugates equals the conjugate of the sum.

6. (a)
$$\begin{aligned}
R &= \sqrt{7^2+10^2} \\
&= \sqrt{149} \\
7\sin\theta - 10\cos\theta &= \sqrt{149}\left(\frac{7}{\sqrt{149}}\sin\theta - \frac{10}{\sqrt{149}}\cos\theta\right) \\
&= R(\sin\theta\cos\alpha - \cos\theta\sin\alpha)
\end{aligned}$$

$$\begin{aligned}
\cos\alpha &= \frac{7}{\sqrt{149}} \\
\alpha &= 0.96 \\
\therefore\quad 7\sin\theta - 10\cos\theta &= \sqrt{149}\sin(\theta-0.96)
\end{aligned}$$

   (b) The minimum value is $-\sqrt{149}$.

$$\begin{aligned}
\sqrt{149}\sin(\theta-0.96) &= -\sqrt{149} \\
\sin(\theta-0.96) &= -1 \\
\theta - 0.96 &= \frac{3\pi}{2} \\
\theta &= \frac{3\pi}{2} + 0.96 \\
&= 5.67
\end{aligned}$$

7. w and $\bar{w}$ must be the vectors in the 1st and 4th quadrants, so $z = -5+3\text{i}$.

$$\begin{aligned}
z^2 &= (-5+3\text{i})^2 \\
&= 25 - 30\text{i} - 9 \\
&= 16 - 30\text{i}
\end{aligned}$$

8. The vector equation of the line through C and D is

$$\begin{aligned}
\mathbf{r} &= \text{C} + \mu\overrightarrow{\text{CD}} \\
&= (-\mathbf{i}-2\mathbf{j}) + \mu(5\mathbf{i}+\mathbf{j}-(-\mathbf{i}-2\mathbf{j})) \\
&= (-\mathbf{i}-2\mathbf{j}) + \mu(6\mathbf{i}+3\mathbf{j})
\end{aligned}$$

   (We use μ as the parameter here because it is independent of λ which has already been used for the other line.)

   Where the two lines intersect,

$$\begin{aligned}
(6\mathbf{i}+12\mathbf{j}) + \lambda(\mathbf{i}-3\mathbf{j}) &= (-\mathbf{i}-2\mathbf{j}) + \mu(6\mathbf{i}+3\mathbf{j}) \\
(6+\lambda)\mathbf{i} + (12-3\lambda)\mathbf{j} &= (-1+6\mu)\mathbf{i} + (-2+3\mu)\mathbf{j} \\
(6+\lambda)\mathbf{i} - (-1+6\mu)\mathbf{i} &= (-2+3\mu)\mathbf{j} - (12-3\lambda)\mathbf{j} \\
(7+\lambda-6\mu)\mathbf{i} &= (-14+3\lambda+3\mu)\mathbf{j}
\end{aligned}$$

   Because $\mathbf{i}$ and $\mathbf{j}$ are not parallel, the only way this equation can be true is if left and right sides both evaluate to the zero vector. This gives us the pair of simultaneous equations that we can solve for λ or μ. (We don't need both as either will allow us to find the point of intersection by substituting it into the corresponding equation of a line, although finding both gives us a cross check that it does in fact yield the same point from both lines.)

$$\begin{aligned}
7+\lambda-6\mu &= 0 \\
-14+3\lambda+3\mu &= 0 \\
-28+6\lambda+6\mu &= 0 \\
-21+7\lambda &= 0 \\
\lambda &= 3 \\
\mathbf{r} &= (6\mathbf{i}+12\mathbf{j}) + 3(\mathbf{i}-3\mathbf{j}) \\
&= 9\mathbf{i}+3\mathbf{j}
\end{aligned}$$

   Check by substituting $\lambda = 3$ into the second equation above:

$$\begin{aligned}
-14+3(3)+3\mu &= 0 \\
-5+3\mu &= 0 \\
\mu &= \frac{5}{3} \\
\mathbf{r} &= (-\mathbf{i}-2\mathbf{j}) + \frac{5}{3}(6\mathbf{i}+3\mathbf{j}) \\
&= (-\mathbf{i}-2\mathbf{j}) + (10\mathbf{i}+5\mathbf{j}) \\
&= 9\mathbf{i}+3\mathbf{j}
\end{aligned}$$

This gives us A$= 9\mathbf{i}+3\mathbf{j}$ so the distance from the origin is

$$\begin{aligned} d &= |\mathrm{A}| \\ &= |9\mathbf{i}+3\mathbf{j}| \\ &= \sqrt{9^2+3^2} \\ &= \sqrt{90} \\ &= 3\sqrt{10} \text{ units} \end{aligned}$$

9. If one of the complex solutions is $x=-2+2\mathrm{i}$ then another must its conjugate $x=-2-2\mathrm{i}$ and one of the quadratic factors will be a multiple of the product of the complex linear factors suggested by these two roots:

$$\begin{aligned} &(x-(-2+2\mathrm{i}))(x-(-2-2\mathrm{i})) \\ &\quad = (x+2-2\mathrm{i})(x+2+2\mathrm{i}) \\ &\quad = x^2+2x+2\mathrm{i}x+2x+4+4\mathrm{i}-2\mathrm{i}x-4\mathrm{i}+4 \\ &\quad = x^2+4x+8 \end{aligned}$$

Hence $f=1$ and $g=4$.

$$\begin{aligned} (5x^2+6x+5)(x^2+4x+8) &= 5x^4+20x^3+40x^2 \\ &\quad +6x^3+24x^2+48x \\ &\quad +5x^2+20x+40 \\ &= 5x^4+26x^3+69x^2+68x+40 \end{aligned}$$

Hence $a=5$, $b=26$, $c=69$, $d=68$ and $e=40$.

Chapter 5

Exercise 5A

1. $\frac{dy}{dx} = 2(4x^{2-1}) = 8x$

2. $\frac{dy}{dx} = 2(3x^{2-1}) + 1(7x^{1-1}) = 6x + 7$

3. $\frac{dy}{dx} = 1(12x^0) - 3(5x^2) = 12 - 15x^2$

4. $\frac{d}{dx}(6x^3) = 3(6x^{3-1}) = 18x^2$

5. $\frac{d}{dx}(6x^3 + 3) = 3(6x^{3-1}) + 0 = 18x^2$

6. $\frac{d}{dx}(3x^3 - x + 1) = 3(3x^{3-1}) - 1(x^{1-1}) + 0 = 9x^2 - 1$

7. $f'(x) = 0$

8. $f'(x) = 2x - 3(4x^2) + 1 = 2x - 12x^2 + 1$

9. $f'(x) = 0 + 1 + 2x + 3x^2 = 1 + 2x + 3x^2$

10.
$$\begin{aligned}\frac{dy}{dx} &= (x-2)\frac{d}{dx}(x+5) + (x+5)\frac{d}{dx}(x-2)\\ &= (x-2) + (x+5)\\ &= 2x+3\end{aligned}$$

11.
$$\begin{aligned}\frac{dy}{dx} &= (2x+3)\frac{d}{dx}(3x+1) + (3x+1)\frac{d}{dx}(2x+3)\\ &= 3(2x+3) + 2(3x+1)\\ &= 6x+9+6x+2\\ &= 12x+11\end{aligned}$$

12.
$$\begin{aligned}\frac{dy}{dx} &= (x^2-5)\frac{d}{dx}(x+7) + (x+7)\frac{d}{dx}(x^2-5)\\ &= (x^2-5) + 2x(x+7)\\ &= x^2 - 5 + 2x^2 + 14x\\ &= 3x^2 + 14x - 5\end{aligned}$$

13.
$$\begin{aligned}\frac{dy}{dx} &= 6x\\ \text{at } x = 2, \quad \frac{dy}{dx} &= 6 \times 2\\ &= 12\end{aligned}$$

14.
$$\begin{aligned}\frac{dy}{dx} &= 6x^2\\ \text{at } x = -1, \quad \frac{dy}{dx} &= 6(-1)^2\\ &= 6\end{aligned}$$

15.
$$\begin{aligned}\frac{dy}{dx} &= 1(x^2-1) + 2x(x-2)\\ &= x^2 - 1 + 2x^2 - 4x\\ &= 3x^2 - 4x - 1\end{aligned}$$
at $x = 3$,
$$\begin{aligned}\frac{dy}{dx} &= 3(3^2) - 4(3) - 1\\ &= 27 - 12 - 1\\ &= 14\end{aligned}$$

16. This question as it stands would be simplest done using the Chain Rule (see the following section in the text). To answer it using only the product rule there are a couple of approaches that could be used. The simplest, and the one appropriate at this stage of learning, is to first simplify and expand the square factor.
$$\begin{aligned}y &= (x+3)(x-2x+1)^2\\ &= (x+3)(-x+1)^2\\ &= (x+3)(x^2-2x+1)\\ \frac{dy}{dx} &= 1(x^2-2x+1) + (2x-2)(x+3)\\ &= x^2 - 2x + 1 + 2x^2 + 6x - 2x - 6\\ &= 3x^2 + 2x - 5\end{aligned}$$
at $x = 2$,
$$\begin{aligned}\frac{dy}{dx} &= 3(2)^2 + 2(2) - 5\\ &= 12 + 4 - 5\\ &= 11\end{aligned}$$

17.
$$\begin{aligned}\frac{dy}{dx} &= 4x\\ 4x &= -8\\ x &= -2\\ y &= 2x^2\\ &= 2(-2)^2\\ &= 8\end{aligned}$$
The curve has a gradient of -8 at $(-2, 8)$.

18.
$$\begin{aligned}\frac{dy}{dx} &= 3x^2 - 7\\ 3x^2 - 7 &= 5\\ 3x^2 &= 12\\ x^2 &= 4\\ x &= \pm 2\end{aligned}$$
At $x = 2$
$$\begin{aligned}y &= (2)^3 - 7(2)\\ &= 8 - 14\\ &= -6\end{aligned}$$
At $x = -2$
$$\begin{aligned}y &= (-2)^3 - 7(-2)\\ &= -8 + 14\\ &= 6\end{aligned}$$
The curve has a gradient of 5 at $(2, -6)$ and $(-2, 6)$

19. $\frac{dy}{dx} = 2x$. At $x = -2$, $\frac{dy}{dx} = 2 \times -2 = -4$. The equation of the tangent line (using the gradient-point form for the equation of a line):

$$\begin{aligned}(y - y_1) &= m(x - x_1)\\ y - 4 &= -4(x - -2)\\ y &= -4(x+2) + 4\\ &= -4x - 8 + 4\\ &= -4x - 4\end{aligned}$$

20. $\frac{dy}{dx} = 5 - 3x^2$. At $x = 1$, $\frac{dy}{dx} = 5 - 3(1)^2 = 2$. The equation of the tangent line is:

$$\begin{aligned}(y - y_1) &= m(x - x_1)\\ y - 4 &= 2(x - 1)\\ y &= 2(x-1) + 4\\ &= 2x - 2 + 4\\ &= 2x + 2\end{aligned}$$

21. (a) $f'(x) = 3 - 6x^2$
    (b) $f'(2) = 3 - 6(2)^2 = 3 - 24 = -21$

22. We expect $\frac{d}{dx}(x^5) = 5x^4$.

$$\begin{aligned}\frac{d}{dx}((x^2)(x^3)) &= 2x(x^3) + 3x^2(x^2)\\ &= 2x^4 + 3x^4\\ &= 5x^4\end{aligned}$$

23. The gradient of the line is 5. The gradient of the curve is

$$\frac{dy}{dx} = 3x^2 - 6x - 4$$

so the $x-$coordinate is the solution to $\frac{dy}{dx} = 5$:

$$\begin{aligned}3x^2 - 6x - 4 &= 5\\ 3x^2 - 6x - 9 &= 0\\ 3(x-3)(x+1) &= 0\\ x &= 3\\ \text{or} \quad x &= -1\end{aligned}$$

For $x = 3$,

$$\begin{aligned}y &= x^3 - 3x^2 - 4x + 1\\ &= 27 - 27 - 12 + 1\\ &= -11\end{aligned}$$

For $x = -1$,

$$\begin{aligned}y &= x^3 - 3x^2 - 4x + 1\\ &= -1 - 3 + 4 + 1\\ &= 1\end{aligned}$$

The curve has the same gradient as the line at $(3, -11)$ and at $(-1, 1)$.

24. For $f(x) = \frac{1}{x}$:

$$\begin{aligned}f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\\ &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}\\ &= \lim_{h\to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h}\\ &= \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h}\\ &= \lim_{h\to 0} \frac{\frac{-h}{x(x+h)}}{h}\\ &= \lim_{h\to 0} \frac{-1}{x(x+h)}\\ &= -\frac{1}{x^2}\end{aligned}$$

This confirms that $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$.
For $f(x) = \sqrt{x}$:

$$\begin{aligned}f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\\ &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}\\ &= \lim_{h\to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)\\ &= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}\\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}\\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}\\ &= \frac{1}{\sqrt{x} + \sqrt{x}}\\ &= \frac{1}{2\sqrt{x}}\end{aligned}$$

This confirms that $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$.

Exercise 5B

1.
$$\begin{aligned}
u &= 2x\\
v &= x-1\\
\frac{\mathrm{d}}{\mathrm{d}x}\frac{2x}{x-1} &= \frac{v\frac{\mathrm{d}u}{\mathrm{d}x}-u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}\\
&= \frac{2(x-1)-1(2x)}{(x-1)^2}\\
&= \frac{2x-2-2x}{x^2-2x+1}\\
&= -\frac{2}{x^2-2x+1}
\end{aligned}$$

2.
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\frac{5x}{2x-3} &= \frac{5(2x-3)-2(5x)}{(2x-3)^2}\\
&= \frac{10x-15-10x}{(2x-3)^2}\\
&= -\frac{15}{(2x-3)^2}
\end{aligned}$$

3.
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\frac{3x}{2x-1} &= \frac{3(2x-1)-2(3x)}{(2x-1)^2}\\
&= \frac{6x-3-6x}{(2x-1)^2}\\
&= -\frac{3}{(2x-1)^2}
\end{aligned}$$

4.
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\frac{3x}{1-5x} &= \frac{3(1-5x)--5(3x)}{(1-5x)^2}\\
&= \frac{3-15x+15x}{(1-5x)^2}\\
&= \frac{3}{(1-5x)^2}
\end{aligned}$$

5.
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\frac{5x+2}{2x-1} &= \frac{5(2x-1)-2(5x+2)}{(2x-1)^2}\\
&= \frac{10x-5-10x-4)}{(2x-1)^2}\\
&= -\frac{9}{(2x-1)^2}
\end{aligned}$$

6.
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\frac{x-6}{5-2x} &= \frac{1(5-2x)--2(x-6)}{(5-2x)^2}\\
&= \frac{5-2x+2x-12}{(5-2x)^2}\\
&= -\frac{7}{(5-2x)^2}
\end{aligned}$$

7.
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\frac{7-3x}{5+2x} &= \frac{-3(5+2x)-2(7-3x)}{(5+2x)^2}\\
&= \frac{-15-6x-14+6x}{(5+2x)^2}\\
&= -\frac{29}{(5+2x)^2}
\end{aligned}$$

8.
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\frac{3x}{x^2-1} &= \frac{3(x^2-1)-2x(3x)}{(x^2-1)^2}\\
&= \frac{3x^2-3-6x^2}{(x^2-1)^2}\\
&= \frac{-3x^2-3}{(x^2-1)^2}\\
&= -\frac{3x^2+3}{(x^2-1)^2}
\end{aligned}$$

9.
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\frac{3x-4}{3x^2+1} &= \frac{3(3x^2+1)-6x(3x-4)}{(3x^2+1)^2}\\
&= \frac{9x^2+3-18x^2+24x}{(3x^2+1)^2}\\
&= \frac{-9x^2+3+24x}{(3x^2+1)^2}\\
&= -\frac{3(3x^2-8x-1)}{(3x^2+1)^2}
\end{aligned}$$

10.
$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\\
&= 5(6x-2)\\
&= 30x-10
\end{aligned}$$

11.
$$\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}t} &= \frac{\mathrm{d}p}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}\\
&= 10s\times -2\\
&= -20(3-2t)\\
&= 40t-60
\end{aligned}$$

12.
$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}x}\\
&= (6u)(2)(2)\\
&= 24u\\
&= 24(2p-1)\\
&= 48p-24\\
&= 48(2x+1)-24\\
&= 96x+24
\end{aligned}$$

13.
$$\begin{aligned}
u &= 2x+3\\
y &= u^3\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\\
&= (3u^2)(2)\\
&= 6(2x+3)^2
\end{aligned}$$

14.
$$\begin{aligned}
u &= 5-3x\\
y &= u^5\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\\
&= (5u^4)(-3)\\
&= -15(5-3x)^4
\end{aligned}$$

15.
$$\begin{aligned}
f'(x) &= 4(3x+5)^3(3)\\
&= 12(3x+5)^3
\end{aligned}$$

16.
$$\begin{aligned} f'(x) &= 4(x+5)^3(1) \\ &= 4(x+5)^3 \end{aligned}$$

17.
$$\begin{aligned} f'(x) &= 7(2x+3)^6(2) \\ &= 14(2x+3)^6 \end{aligned}$$

18.
$$\begin{aligned} f'(x) &= 3(5x^2+2)^2(10x) \\ &= 30x(5x^2+2)^2 \end{aligned}$$

19.
$$\begin{aligned} f'(x) &= 3(1-2x)^2(-2) \\ &= -6(1-2x)^2 \end{aligned}$$

20.
$$\begin{aligned} f'(x) &= 5+5(4x+1)^4(4) \\ &= 5+20(4x+1)^4 \end{aligned}$$

21.
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= (x-3)^5\frac{\mathrm{d}}{\mathrm{d}x}(x^2) + x^2\frac{\mathrm{d}}{\mathrm{d}x}(x-3)^5 \\ &= 2x(x-3)^5 + x^2(5(x-3)^4) \\ &= 2x(x-3)(x-3)^4 + 5x^2(x-3)^4 \\ &= (2x^2-6x)(x-3)^4 + 5x^2(x-3)^4 \\ &= (7x^2-6x)(x-3)^4 \end{aligned}$$

22.
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= (x+1)^3\frac{\mathrm{d}}{\mathrm{d}x}(3x) + 3x\frac{\mathrm{d}}{\mathrm{d}x}(x+1)^3 \\ &= 3(x+1)^3 + 3x(3(x+1)^2) \\ &= 3((x+1)(x+1)^2 + 3x(x+1)^2) \\ &= 3(4x+1)(x+1)^2 \end{aligned}$$

23.
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= (x^2+3)^4\frac{\mathrm{d}}{\mathrm{d}x}(2x) + 2x\frac{\mathrm{d}}{\mathrm{d}x}(x^2+3)^4 \\ &= 2(x^2+3)^4 + 2x(4(x^2+3)^3(2x)) \\ &= 2(x^2+3)^4 + 16x^2(x^2+3)^3 \\ &= 2(x^2+3)(x^2+3)^3 + 16x^2(x^2+3)^3 \\ &= (2x^2+6+16x^2)(x^2+3)^3 \\ &= 6(3x^2+1)(x^2+3)^3 \end{aligned}$$

24.
$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}((5x-1)(x+5)) \\ &= (x+5)\frac{\mathrm{d}}{\mathrm{d}x}(5x-1) + (5x-1)\frac{\mathrm{d}}{\mathrm{d}x}(x+5) \\ &= 5(x+5) + (5x-1) \\ &= 5x+25+5x-1 \\ &= 10x+24 \end{aligned}$$

25.
$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\frac{2x+3}{3x+2} \\ &= \frac{(3x+2)\frac{\mathrm{d}}{\mathrm{d}x}(2x+3) - (2x+3)\frac{\mathrm{d}}{\mathrm{d}x}(3x+2)}{(3x+2)^2} \\ &= \frac{2(3x+2) - 3(2x+3)}{(3x+2)^2} \\ &= \frac{6x+4-6x-9}{(3x+2)^2} \\ &= -\frac{5}{(3x+2)^2} \end{aligned}$$

26.
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(3x^2-1)^4 &= 4(3x^2-1)^3\frac{\mathrm{d}}{\mathrm{d}x}(3x^2-1) \\ &= 4(3x^2-1)^3(6x) \\ &= 24x(3x^2-1)^3 \end{aligned}$$

27.
$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}(2x^2-3x+1)^3 \\ &\quad= 3(2x^2-3x+1)^2\frac{\mathrm{d}}{\mathrm{d}x}(2x^2-3x+1) \\ &\quad= 3(2x^2-3x+1)^2(4x-3) \\ &\quad= 3(4x-3)(2x^2-3x+1)^2 \end{aligned}$$

28. The quotient rule might seem the obvious approach to this one, but it's easier to simplify before differentiating:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3+5x}{x} &= \frac{\mathrm{d}}{\mathrm{d}x}(x^2+5) \\ &= 2x \end{aligned}$$

Does the quotient rule give the same answer?

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3+5x}{x} &= \frac{x\frac{\mathrm{d}}{\mathrm{d}x}(x^3+5x) - (x^3+5x)\frac{\mathrm{d}}{\mathrm{d}x}(x)}{x^2} \\ &= \frac{x(3x^2+5) - (x^3+5x)}{x^2} \\ &= \frac{3x^3+5x-x^3-5x}{x^2} \\ &= \frac{2x^3}{x^2} \\ &= 2x \end{aligned}$$

29.
$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2+4x+3}{x+1} \\ &= \frac{(x+1)\frac{\mathrm{d}}{\mathrm{d}x}(x^2+4x+3) - (x^2+4x+3)\frac{\mathrm{d}}{\mathrm{d}x}(x+1)}{(x+1)^2} \\ &= \frac{(x+1)(2x+4) - (x^2+4x+3)}{(x+1)^2} \\ &= \frac{2x^2+4x+2x+4-x^2-4x-3}{(x+1)^2} \\ &= \frac{x^2+2x+1}{x^2+2x+1} \\ &= 1 \end{aligned}$$

(This would be simpler if you realised that $\frac{x^2+4x+3}{x+1} = \frac{(x+3)(x+1)}{x+1}$ then simplify before differentiating.)

30.
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 4(5-2x)^3(-2) \\ &= -8(5-2x)^3 \end{aligned}$$

At $x = 2$ this evaluates to

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= -8(5-2(2))^3 \\ &= -8(1)^3 \\ &= -8 \end{aligned}$$

31.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4(x-3)-4x}{(x-3)^2}\\ &= \frac{4x-12-4x}{(x-3)^2}\\ &= -\frac{12}{(x-3)^2}\end{aligned}$$

At $x = 5$ this evaluates to

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{12}{((5)-3)^2}\\ &= -\frac{12}{2^2}\\ &= -3\end{aligned}$$

32.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{7x^6(x^2)-2x(x^7)}{(x^2)^2}\\ &= \frac{7x^8-2x^8}{x^4}\\ &= \frac{5x^8}{x^4}\\ &= 5x^4\end{aligned}$$

(just as expected.)

33. The gradient function is

$$\begin{aligned}y' &= 3(2x-5)^2(2)\\ &= 6(2x-5)^2\end{aligned}$$

and at $x = 2$ this evaluates to

$$\begin{aligned}y' &= 6(2(2)-5)^2\\ &= 6(-1)^2\\ &= 6\end{aligned}$$

The gradient-point form for the equation of a line:

$$\begin{aligned}(y-y_1) &= m(x-x_1)\\ y-(-1) &= 6(x-2)\\ y+1 &= 6x-12\\ y &= 6x-13\end{aligned}$$

34. Where the curve and line intersect,

$$\begin{aligned}\frac{5x^2}{x-1} &= 5x+3\\ 5x^2 &= (x-1)(5x+3)\\ &= 5x^2+3x-5x-3\\ 0 &= -2x-3\\ 2x &= -3\\ x &= -1.5\\ y &= 5x+3\\ &= 5(-1.5)+3\\ &= -4.5\end{aligned}$$

The line and curve intersect at $(-1.5, -4.5)$.

The gradient function of the curve is

$$y' = \frac{(x-1)(10x)-(5x^2)(1)}{(x-1)^2)}$$

At $x = -1.5$ this evaluates to

$$\begin{aligned}y' &= \frac{(-1.5-1)(10)(-1.5)-(5(-1.5)^2)(1)}{(-1.5-1)^2}\\ &= \frac{37.5-11.25}{(-2.5)^2}\\ &= \frac{26.25}{6.25}\\ &= \frac{105}{25}\\ &= 4.2\end{aligned}$$

35. The gradient function is

$$\begin{aligned}y' &= \frac{(x-2)(2x)-(x^2)(1)}{(x-2)^2}\\ &= \frac{2x^2-4x-x^2}{(x-2)^2}\\ &= \frac{x^2-4x}{(x-2)^2}\end{aligned}$$

At $x = 3$ this evaluates to

$$\begin{aligned}y' &= \frac{3^2-4\times 3}{(3-2)^2}\\ &= -3\end{aligned}$$

The gradient m of the normal is given by

$$\begin{aligned}-3m &= -1\\ m &= \frac{1}{3}\end{aligned}$$

Then using the gradient-point form to find the equation of the normal

$$\begin{aligned}(y-y_1) &= m(x-x_1)\\ y-9 &= \frac{1}{3}(x-3)\\ y &= \frac{x}{3}+8\end{aligned}$$

36. The gradient function is

$$\begin{aligned}y' &= (2x-5)^3(2)+(2x-1)(3(2x-5)^2(2))\\ &= 2(2x-5)^3+6(2x-1)(2x-5)^2\end{aligned}$$

Factorising:

$$\begin{aligned}y' &= 2(2x-5)^2((2x-5)+3(2x-1))\\ &= 2(2x-5)^2(2x-5+6x-3)\\ &= 2(2x-5)^2(8x-8)\\ &= 16(2x-5)^2(x-1)\end{aligned}$$

Thus the gradient is zero where

$$\begin{aligned} 2x-5&=0\\ x&=2.5\\ \text{or } x-1&=0\\ x&=1 \end{aligned}$$

Substituting these values back into the original equation to find their corresponding y values:

$$\begin{aligned} y&=(2(2.5)-1)(2(2.5)-5)^3\\ &=4\times 0^3\\ &=0\\ \text{or } y&=(2(1)-1)(2(1)-5)^3\\ &=1\times(-3)^3\\ &=-27 \end{aligned}$$

The gradient of the curve is zero at $(2.5, 0)$ and at $(1, -27)$.

37. The gradient function is

$$\begin{aligned} y'&=\frac{(2x+1)(2x+2)-(x^2+2x+3)(2)}{(2x+1)^2}\\ &=\frac{(4x^2+4x+2x+2)-(2x^2+4x+6)}{(2x+1)^2}\\ &=\frac{2x^2+2x-4}{(2x+1)^2}\\ &=\frac{2(x^2+x-2)}{(2x+1)^2}\\ &=\frac{2(x+2)(x-1)}{(2x+1)^2} \end{aligned}$$

Thus the gradient is zero where

$$\begin{aligned} x+2&=0\\ x&=-2\\ \text{or } x-1&=0\\ x&=1 \end{aligned}$$

Substituting these values back into the original equation to find their corresponding y values:

$$\begin{aligned} y&=\frac{(-2)^2+2(-2)+3}{2(-2)+1}\\ &=-1\\ \text{or } y&=\frac{(1)^2+2(1)+3}{2(1)+1}\\ &=2 \end{aligned}$$

The gradient of the curve is zero at $(-2, -1)$ and at $(1, 2)$.

38. (a) First, find a by substituting $x=-3$ into the equation of the curve:

$$\begin{aligned} a&=\frac{5(-3)-7}{2(-3)+10}\\ &=\frac{-22}{4}\\ &=-5.5 \end{aligned}$$

Now b is the gradient of the tangent line at $(-3, -5.5)$ and hence the gradient of the curve at that point, so we can find b by substituting $x=-3$ into the gradient function.

$$\begin{aligned} y'&=\frac{(2x+10)(5)-(5x-7)(2)}{(2x+10)^2}\\ &=\frac{10x+50-10x+14}{(2x+10)^2}\\ &=\frac{64}{(2x+10)^2}\\ b&=\frac{64}{(2(-3)+10)^2}\\ &=\frac{64}{4^2}\\ &=4 \end{aligned}$$

The equation of the tangent line is

$$\begin{aligned} y-y_1&=m(x-x_1)\\ y-a&=b(x--3)\\ y--5.5&=4(x+3)\\ y+5.5&=4x+12\\ y&=4x+6.5 \end{aligned}$$

hence $c=6.5$.

(b) Solve $y'=b=4$ (already knowing that $x=-3$ is one solution):

$$\begin{aligned} y'&=4\\ \frac{64}{(2x+10)^2}&=4\\ \frac{16}{(2x+10)^2}&=1\\ 16&=(2x+10)^2\\ 2x+10&=\pm4\\ 2x&=-10\pm4\\ x&=-5\pm2\\ x&=-7\\ \text{or}\quad x&=-3 \end{aligned}$$

Substituting $x=-7$ into the original equation:

$$\begin{aligned} y&=\frac{5(-7)-7}{2(-7)+10}\\ &=\frac{-42}{-4}\\ &=10.5 \end{aligned}$$

The other point where the tangent to the curve is parallel to $y=bx+3$ has coordinates $(-7, 10.5)$.

39. (a) First, find a by substituting $x = 3$ and $y' = 2$ into the gradient equation and solving. (Remember, a is a constant, so its derivative is zero.)

$$\begin{aligned}
y' &= \frac{(2x-11)(-6)-(a-6x)(2)}{(2x-11)^2}\\
&= \frac{-12x+66-2a+12x}{(2x-11)^2}\\
&= \frac{66-2a}{(2x-11)^2}\\
2 &= \frac{66-2a}{(2(3)-11)^2}\\
2 &= \frac{66-2a}{(-5)^2}\\
2 &= \frac{66-2a}{25}\\
66-2a &= 50\\
-2a &= -16\\
a &= 8
\end{aligned}$$

Now substitute this and $x = 3$ into the original equation to find b:

$$\begin{aligned}
b &= \frac{8-6(3)}{2(3)-11}\\
&= \frac{-10}{-5}\\
&= 2
\end{aligned}$$

(b) Solve $y' = 2$ (already knowing that $x = 3$ is one solution):

$$\begin{aligned}
y' &= 2\\
\frac{66-2a}{(2x-11)^2} &= 2
\end{aligned}$$

Substituting $a = 8$:

$$\begin{aligned}
\frac{50}{(2x-11)^2} &= 2\\
\frac{25}{(2x-11)^2} &= 1\\
25 &= (2x-11)^2\\
2x-11 &= \pm 5\\
2x &= 11 \pm 5\\
2x &= 16\\
x &= 8\\
\text{or}\quad 2x &= 6\\
x &= 3
\end{aligned}$$

Substituting $x = 8$ and $a = 8$ into the original equation:

$$\begin{aligned}
y &= \frac{8-6(8)}{2(8)-11}\\
&= \frac{-40}{5}\\
&= -8
\end{aligned}$$

The other point where the curve has a gradient of 2 is at $(8, -8)$.

40. From the first curve:

$$\begin{aligned}
y' &= (2x-3)^3(1) + (x+1)(3(2x-3)^2(2))\\
&= (2x-3)^3 + 6(x+1)(2x-3)^2\\
&= (2x-3)^2(2x-3+6x+6)\\
&= (2x-3)^2(8x+3)
\end{aligned}$$

At $x = 2$:

$$\begin{aligned}
c &= (2(2)-3)^2(8(2)+3)\\
&= 19
\end{aligned}$$

Thus the gradient of all three curves at $x = 2$ is 19.

From the second curve:

$$\begin{aligned}
y' &= 6x - (-a(x-1)^{-2})\\
&= 6x + \frac{a}{(x-1)^2}
\end{aligned}$$

At $x = 2$:

$$\begin{aligned}
19 &= 6(2) + \frac{a}{(2-1)^2}\\
&= 12 + a\\
a &= 7
\end{aligned}$$

From the third curve:

$$\begin{aligned}
y' &= \frac{(4-x)(b)-(bx+12)(-1)}{(4-x)^2}\\
&= \frac{4b-bx+bx+12}{(4-x)^2}\\
&= \frac{4b+12}{(4-x)^2}
\end{aligned}$$

At $x = 2$:

$$\begin{aligned}
19 &= \frac{4b+12}{(4-2)^2}\\
&= b+3\\
b &= 16
\end{aligned}$$

Miscellaneous Exercise 5

1. (a) Use the null factor law to give $x = 3$ or $x = -7$.

   (b)
$$\begin{aligned} 2x - 5 &= 0 & \text{or} \quad 4x + 1 &= 0 \\ 2x &= 5 & 4x &= -1 \\ x &= 2.5 & x &= -0.25 \end{aligned}$$

   (c) First factorise then use the null factor law:
$$\begin{aligned} (x-4)(x+3) &= 0 \\ x &= 4 \\ \text{or} \quad x &= -3 \end{aligned}$$

   (d)
$$\begin{aligned} (x+7)(x-2) &= 0 \\ x &= -7 \\ \text{or} \quad x &= 2 \end{aligned}$$

   (e)
$$\begin{aligned} 5(x^2 + x - 12) &= 0 \\ 5(x+4)(x-3) &= 0 \\ x &= -4 \\ \text{or} \quad x &= 3 \end{aligned}$$

   (f)
$$\begin{aligned} 4(x^2 + 9x - 10) &= 0 \\ 4(x+10)(x-1) &= 0 \\ x &= -10 \\ \text{or} \quad x &= 1 \end{aligned}$$

2. LHS:
$$\begin{aligned} \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} &= \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} \times \frac{\cos\theta - \sin\theta}{\cos\theta - \sin\theta} \\ &= \frac{\cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta}{\cos^2\theta - \sin^2\theta} \\ &= \frac{1 - 2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} \\ &= \frac{1 - \sin 2\theta}{\cos 2\theta} \end{aligned}$$
□

3. From the null factor law, using the first factor:
$$\begin{aligned} 6 + 25\sin\theta &= 0 \\ \sin\theta &= -\frac{6}{25} \\ &= -0.24 \end{aligned}$$
sine is negative in the 3rd and 4th quadrants
$$\begin{aligned} \theta &= \pi + \frac{\pi}{13} \\ &= \frac{14\pi}{13} \\ \text{or} \quad \theta &= 2\pi - \frac{\pi}{13} \\ &= \frac{25\pi}{13} \end{aligned}$$
using the second factor:
$$\begin{aligned} 1 - 2\cos\theta &= 0 \\ \cos\theta &= 0.5 \end{aligned}$$
cosine is positive in the 1st and 4th quadrants
$$\begin{aligned} \theta &= \frac{\pi}{3} \\ \text{or} \quad \theta &= \frac{5\pi}{3} \end{aligned}$$

4. (a) O is the midpoint of AB, so it has coordinates:
$$\left(\frac{1+9}{2}, \frac{2+-4}{2}\right) = (5, -1)$$

   (b) The radius is the distance OA:
$$r = \sqrt{(1-5)^2 + (2 - -1)^2} = 5$$

   (c) The vector equation of a circle radius 5 centred at $(5, -1)$ is
$$\left|\mathbf{r} - \begin{pmatrix} 5 \\ -1 \end{pmatrix}\right| = 5$$

5. If the equation is not to have complex solutions, $b^2 - 4ac$ must be non-negative:
$$\begin{aligned} (-q)^2 - 4(4)(3) &\geq 0 \\ q^2 - 48 &\geq 0 \\ q^2 &\geq 48 \\ q &\geq 4\sqrt{3} \\ \text{or} \quad q &\leq -4\sqrt{3} \end{aligned}$$

6. (a)
$$\begin{aligned} z + w &= -5 + 2\mathrm{i} + -3\mathrm{i} \\ &= -5 - \mathrm{i} \end{aligned}$$

   (b)
$$\begin{aligned} zw &= (-5 + 2\mathrm{i})(-3\mathrm{i}) \\ &= 15\mathrm{i} + 6 \\ &= 6 + 15\mathrm{i} \end{aligned}$$

   (c)
$$\bar{z} = -5 - 2\mathrm{i}$$

   (d)
$$\begin{aligned} \bar{z}\bar{w} &= (-5 - 2\mathrm{i})(3\mathrm{i}) \\ &= -15\mathrm{i} + 6 \\ &= 6 - 15\mathrm{i} \end{aligned}$$

   (e)
$$\begin{aligned} z^2 &= (-5 + 2\mathrm{i})^2 \\ &= 25 - 20\mathrm{i} - 4 \\ &= 21 - 20\mathrm{i} \end{aligned}$$

   (f)
$$\begin{aligned} (zw)^2 &= (6 + 15\mathrm{i})^2 \\ &= 36 + 180\mathrm{i} - 225 \\ &= -189 + 180\mathrm{i} \end{aligned}$$

   (g)
$$\begin{aligned} p &= \mathrm{Re}(\bar{z}) + \mathrm{Im}(\bar{w})\mathrm{i} \\ &= \mathrm{Re}(z) - \mathrm{Im}(w)\mathrm{i} \\ &= -5 + 3\mathrm{i} \end{aligned}$$

7. Let $z = a + b\mathrm{i}$

$$\begin{aligned} 5z - \bar{z} &= -8 + 12\mathrm{i} \\ 5(a + b\mathrm{i}) - (a - b\mathrm{i}) &= -8 + 12\mathrm{i} \\ 4a + 6b\mathrm{i} &= -8 + 12\mathrm{i} \\ a &= -2 \\ b &= 2 \\ z &= -2 + 2\mathrm{i} \end{aligned}$$

8. $$\begin{aligned} (x + \mathrm{i}y)^2 &= 96 - 40\mathrm{i} \\ x^2 + 2xy\mathrm{i} - y^2 &= 96 - 40\mathrm{i} \end{aligned}$$

From the imaginary components:

$$\begin{aligned} 2xy &= -40 \\ y &= -\frac{20}{x} \end{aligned}$$

From the real components:

$$\begin{aligned} x^2 - y^2 &= 96 \\ x^2 - (-\frac{20}{x})^2 &= 96 \\ x^2 - \frac{400}{x^2} &= 96 \\ x^4 - 400 &= 96x^2 \\ x^4 - 96x^2 - 400 &= 0 \\ (x^2 - 100)(x^2 + 4) &= 0 \end{aligned}$$

The second factor has no real solutions, so we can disregard it and focus on the first.

$$\begin{aligned} x^2 - 100 &= 0 \\ x^2 &= 100 \\ x &= \pm 10 \\ y &= -\frac{20}{\pm 10} \\ &= \mp 2 \end{aligned}$$

(x, y) is $(10, -2)$ or $(-10, 2)$

9. (a) $$\begin{aligned} (2y - 1)(y + 1) &= 2y^2 + 2y - y - 1 \\ &= 2y^2 + y - 1 \end{aligned}$$

(b) $$\begin{aligned} 1 + \sin x &= 2\cos^2 x \\ &= 2(1 - \sin^2 x) \\ &= 2 - 2\sin^2 x \end{aligned}$$

$$\begin{aligned} 2\sin^2 x + \sin x - 1 &= 0 \\ (2\sin x - 1)(\sin x + 1) &= 0 \end{aligned}$$

From the first factor:

$$\begin{aligned} 2\sin x - 1 &= 0 \\ \sin x &= \frac{1}{2} \\ x &= \frac{\pi}{6}; \frac{5\pi}{6}; -\frac{7\pi}{6}; \text{ or } -\frac{11\pi}{6} \end{aligned}$$

From the second factor:

$$\begin{aligned} \sin x + 1 &= 0 \\ \sin x &= -1 \\ x &= \frac{3\pi}{2}; \text{ or } -\frac{\pi}{2} \end{aligned}$$

10. The displacement vector from ship to yacht is

$$\begin{aligned} {}_{\text{YACHT}}\mathbf{r}_{\text{SHIP}} &= \mathbf{r}_{\text{YACHT}} - \mathbf{r}_{\text{SHIP}} \\ &= (9\mathbf{i} + 8\mathbf{j}) - (10\mathbf{i} + 5\mathbf{j}) \\ &= (-\mathbf{i} + 3\mathbf{j})\text{km} \end{aligned}$$

The velocity of the ship relative to the yacht is

$$\begin{aligned} {}_{\text{SHIP}}\mathbf{v}_{\text{YACHT}} &= \mathbf{v}_{\text{SHIP}} - \mathbf{v}_{\text{YACHT}} \\ &= (8\mathbf{i} + 7\mathbf{j}) - (12\mathbf{i} - 5\mathbf{j}) \\ &= (-4\mathbf{i} + 12\mathbf{j})\text{km/h} \end{aligned}$$

Since ${}_{\text{YACHT}}\mathbf{r}_{\text{SHIP}} = 0.25\,{}_{\text{SHIP}}\mathbf{v}_{\text{YACHT}}$, the ships will collide in a quarter of an hour, i.e. at 9:15am.

The position of the collision is

$$\begin{aligned} \mathbf{r} &= (10\mathbf{i} + 5\mathbf{j}) + 0.25(8\mathbf{i} + 7\mathbf{j}) \\ &= (12\mathbf{i} + 6.75\mathbf{j})\text{km} \end{aligned}$$

11. (a) The conjugate of w has the same real component and the opposite imaginary component: it's a reflection in the $x-$axis. Diagram B.

(b) If $z + w$ is real, then they must have opposite imaginary components. This is true for diagrams B and D.

(c) If zw is real then $\text{Re}(z) \times \text{Im}(w) + \text{Im}(z) \times \text{Re}(w) = 0$ (since the other terms that arise from the multiplication are real).

$$\begin{aligned} \text{Re}(z) \times \text{Im}(w) + \text{Im}(z) \times \text{Re}(w) &= 0 \\ \text{Re}(z) \times \text{Im}(w) &= -\text{Im}(z) \times \text{Re}(w) \\ \frac{\text{Im}(w)}{\text{Re}(w)} &= -\frac{\text{Im}(z)}{\text{Re}(z)} \end{aligned}$$

On the Argand diagram this represents points having the opposite gradient. This is true for diagrams A, B and F.

(d) Numbers with an imaginary part of 1 are shown in diagrams A and C.

(e) Numbers having the absolute value of their imaginary part equal to 1 are shown in diagrams A, B, C and D.

(f) Since w has a positive imaginary component, this is no different from part (d) above: diagrams A and C.

(g) This results in an imaginary part equal to $\text{Re}(w)$ and real part equal to $-\text{Im}(w)$, i.e. a 90° rotation. This is shown in diagram E.

(h) If we multiply $\frac{\bar{w}}{z}$ by $\frac{\bar{z}}{\bar{z}}$ the denominator will always be real, so $\frac{\bar{w}}{z}$ is real if $\bar{w}\bar{z}$ is real. This is similar to part (c) above with a similar result:

$$\begin{aligned}\mathrm{Re}(\bar{w}) \times \mathrm{Im}(\bar{z}) + \mathrm{Im}(\bar{w}) \times \mathrm{Re}(\bar{z}) &= 0\\ \mathrm{Re}(w) \times -\mathrm{Im}(z) + -\mathrm{Im}(w) \times \mathrm{Re}(z) &= 0\\ -\mathrm{Re}(w) \times \mathrm{Im}(z) &= \mathrm{Im}(w) \times \mathrm{Re}(z)\\ \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} &= -\frac{\mathrm{Im}(w)}{\mathrm{Re}(w)}\end{aligned}$$

On the Argand diagram this represents points having the opposite gradient, just as in part (c). This is true for diagrams A, B and F.

12. (a) The radius is 5. The centre has position vector $7\mathbf{i}-\mathbf{j}$ which corresponds to Cartesian coordinates $(7, -1)$.

(b) The radius is 6. $|\mathbf{r} - 7\mathbf{i} - \mathbf{j}| = |\mathbf{r} - (7\mathbf{i} + \mathbf{j})|$ The centre has position vector $7\mathbf{i} + \mathbf{j}$ which corresponds to Cartesian coordinates $(7, 1)$.

(c) The radius is $\sqrt{18} = 3\sqrt{2}$. The centre is the origin, $(0, 0)$.

(d) The radius is $\sqrt{75} = 5\sqrt{3}$. The centre is $(1, -8)$.

(e)
$$\begin{aligned}x^2 + y^2 + 2x &= 14y + 50\\ x^2 + y^2 + 2x - 14y &= 50\\ (x+1)^2 - 1 + (y-7)^2 - 49 &= 50\\ (x+1)^2 + (y-7)^2 &= 100\end{aligned}$$

The radius is $\sqrt{100} = 10$. The centre is $(-1, 7)$.

(f)
$$\begin{aligned}x^2 + 10x + y^2 &= 151 + 14y\\ x^2 + 10x + y^2 - 14y &= 151\\ (x+5)^2 - 25 + (y-7)^2 - 49 &= 151\\ (x+5)^2 + (y-7)^2 &= 225\end{aligned}$$

The radius is $\sqrt{225} = 15$. The centre is $(-5, 7)$.

13. (a)
$$\begin{aligned}&3x^3 - 11x^2 + 25x - 25\\ &= (ax-b)(x^2 + cx + 5)\\ &= ax^3 + acx^2 + 5ax - bx^2 - bcx - 5b\\ &= ax^3 + (ac-b)x^2 + (5a - bc)x - 5b\end{aligned}$$

From the x^3 term:

$$a = 3$$

From the constant term:

$$\begin{aligned}-5b &= -25\\ b &= 5\end{aligned}$$

From the x^2 term:

$$\begin{aligned}ac - b &= -11\\ 3c - 5 &= -11\\ 3c &= -6\\ c &= -2\end{aligned}$$

(b) Use the results from (a) to factor the expression

$$\begin{aligned}3x^3 - 11x^2 + 25x - 25 &= 0\\ (3x-5)(x^2 - 2x + 5) &= 0\end{aligned}$$

From the linear factor:

$$\begin{aligned}3x - 5 &= 0\\ x &= \frac{5}{3}\end{aligned}$$

From the quadratic factor, using the quadratic formula:

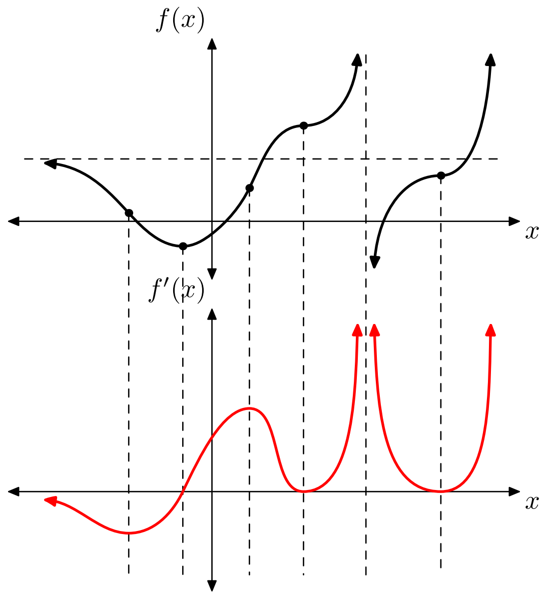
$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\ &= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)}\\ &= \frac{2 \pm \sqrt{4 - 20}}{2}\\ &= \frac{2 \pm \sqrt{-16}}{2}\\ &= \frac{2 \pm 4\mathrm{i}}{2}\\ &= 1 \pm 2\mathrm{i}\end{aligned}$$

14. (a)
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 4(2x^3 - 5)^3(6x^2)\\ &= 24x^2(2x^3 - 5)^3\end{aligned}$$

(b)
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (2x+1)^3(6x) + (3x^2+2)(3(2x+1)^2(2))\\ &= 6x(2x+1)^3 + 6(3x^2+2)(2x+1)^2\\ &= 6(2x+1)^2(x(2x+1) + (3x^2+2))\\ &= 6(2x+1)^2(2x^2 + x + 3x^2 + 2)\\ &= 6(2x+1)^2(2x^2 + x + 3x^2 + 2)\\ &= 6(2x+1)^2(5x^2 + x + 2)\end{aligned}$$

15. Working left to right:

- The curve begins with a small negative gradient
- Gradient decreases to a minimum at the first marked point
- At the second marked point the curve is horizontal, so the gradient is zero. After this the gradient continues to increase.
- At the third point the gradient reaches its local maximum and begins decreasing.
- At the fourth point the curve is horizontal, so the gradient is zero.
- As it approaches the vertical asymptote the gradient of the curve increases.
- On the other side of the asymptote the gradient decreases to zero at the last marked point, then increases again.



Chapter 6

Exercise 6A

1. Vertical asymptote when the denominator is zero: $x = 0$.
2. Vertical asymptote when the denominator is zero: $x = 1$.
3. Vertical asymptote when the denominator is zero: $x = 3$ or $x = \frac{1}{2}$.
4. Vertical asymptote when the denominator is zero: $x = 3$.
5. x^2 is always non-negative (for x real) so $y \not< 0$.
6. The square root is non-negative. The function is defined for all x such that $x - 3$ is non-negative; this has no upper limit so there is no upper limit on y: $y \not< 0$.
7. $y \neq 0$
8. $\frac{3}{x} \neq 0$ so $y = 2 + \frac{3}{x} \neq 2$
9. $y \neq 0$
10. As x increases without bound, or decreases without bound, y approaches 1, but since the numerator and denominator can never be equal, $y \neq 1$.
11. As $x \to +\infty$ the x^2 term dominates, so $y \to +\infty$.
    As $x \to -\infty$ the x^2 term dominates, so $y \to +\infty$.
12. As $x \to +\infty$ the $-2x^3$ term dominates, so $y \to -\infty$.
    As $x \to -\infty$ the $-2x^3$ term dominates, so $y \to +\infty$.
13. As $x \to +\infty$ the x term in the denominator dominates, so $y \to 0$.
    As $x \to -\infty$ the x term in the denominator dominates, so $y \to 0$.
14. As $x \to +\infty$ the x term in the numerator and denominator dominates, so $y \to 1$.
    As $x \to -\infty$ the x term in the numerator and denominator dominates, so $y \to 1$.
15. As $x \to +\infty$ the $5x^2$ term in the numerator and the x^2 term in the denominator dominates, so $y \to 5$.
    As $x \to -\infty$ the $5x^2$ term in the numerator and the x^2 term in the denominator dominates, so $y \to 5$.
16. As $x \to +\infty$ the $3x^2$ term in the numerator and the x^2 term in the denominator dominates, so $y \to 3$.
    As $x \to -\infty$ the $3x^2$ term in the numerator and the x^2 term in the denominator dominates, so $y \to 3$.
17. For $x > 0$, $\frac{1}{x} > 0$ so as $x \to 0^+$ then $y \to +\infty$.
    For $x < 0$, $\frac{1}{x} < 0$ so as $x \to 0^-$ then $y \to -\infty$.
18. For $x > 3$, $x - 3 > 0$ so as $x \to 3^+$ then $y \to +\infty$.
    For $x < 3$, $x - 3 < 0$ so as $x \to 3^-$ then $y \to -\infty$.
19. For $x > 1$, $1 - x < 0$ so as $x \to 1^+$ then $y \to -\infty$.
    For $x < 1$, $1 - x > 0$ so as $x \to 1^-$ then $y \to +\infty$.
20. As $x \to 0$ the $\frac{1}{x^2}$ term dominates.
    For $x > 0$, $\frac{1}{x^2} > 0$ so as $x \to 0^+$ then $y \to +\infty$.
    For $x < 0$, $\frac{1}{x^2} > 0$ so as $x \to 0^-$ then $y \to +\infty$.
21. (a) Key features: $y \to +\infty$ as $x \to 3$ from above or from below. We expect a denominator that goes to 0 only when $x \to 3$ but that is positive either side of $x = 3$ so the denominator must be raised to an even power. The only equation that matches is $y = \frac{1}{(x-3)^2}$.
    (b) Key features: y goes to infinity as x approaches 3 or -3. For $x > 3$, $y > 0$. The equation that matches is $y = \frac{1}{(x+3)(x-3)}$.
    (c) Key features: $y \to +\infty$ as $x \to 3^+$ and $y \to -\infty$ as $x \to 3^-$. We expect a denominator that goes to 0 only when $x \to 3$ and the expression is positive for $x > 3$. The only equation that matches is $y = \frac{1}{(x-3)}$.

Exercise 6B

1. (a) Third piece: $f(5) = 3 \times 5 = 15$
   (b) First piece: $f(-2) = -2$
   (c) Second piece: $f(3) = 3^2 = 9$
   (d) First piece: $f(-4) = -4$
   (e) Second piece: $f(2.5) = (2.5)^2 = 6.25$
2. (a) Third piece: $f(3) = 2 \times 3 = 6$
   (b) First piece: $f(0) = 0 + 1 = 1$
   (c) Second piece: $f(1) = 3$
   (d) First piece: $f(-4) = -4 + 1 = -3$
   (e) Third piece: $f(3.5) = 2(3.5) = 7$

3. (a) Not a function for domain $\Re$ because $\mathrm{f}(0)$ has two values ($2(0)+3=3$ and 1) where the two pieces overlap, so it fails the vertical line test at $x=0$.
   (b) Not a function for domain $\Re$ because $\mathrm{f}(x)$ is not defined for $x<-3$.
   (c) Not a function for domain $\Re$ because $\mathrm{f}(x)$ is not defined for $7<x<8$. (The first and second pieces overlap at $x=4$, but this is not a problem because both pieces give $\mathrm{f}(4)=8$.)
   (d) Not a function because $\mathrm{f}(5)=\pm 2$: it fails the vertical line test at $x=5$.

4. The key point for $|x-1|$ is at $x=1$. This gives us
$$\mathrm{f}(x)=\begin{cases}-(x-1) & \text{for } x<1\\ x-1 & \text{for } x\geq 1\end{cases}$$
that simplifies to
$$\mathrm{f}(x)=\begin{cases}1-x & \text{for } x<1\\ x-1 & \text{for } x\geq 1\end{cases}$$

5. The key point for $|2x-5|$ is at $x=2.5$. This gives us
$$\mathrm{f}(x)=\begin{cases}-(2x-5) & \text{for } x<2.5\\ 2x-5 & \text{for } x\geq 2.5\end{cases}$$
that simplifies to
$$\mathrm{f}(x)=\begin{cases}5-2x & \text{for } x<2.5\\ 2x-5 & \text{for } x\geq 2.5\end{cases}$$

6. Working left to right, the first part (for $x<-2$) has equation $y=x+4$. The second part (for $-2\leq x<2$) has equation $y=x^2-2$. The third part (for $x\geq 2$) has equation $y=-x+4$. This gives
$$\mathrm{f}(x)=\begin{cases}x+4 & \text{for } x<-2\\ x^2-2 & \text{for } -2\leq x<2\\ -x+4 & \text{for } x\geq 2\end{cases}$$

7. Working left to right, the first part (for $x<-2$) has equation $y=4$. The second part (for $-2\leq x<1$) has equation $y=x^2$. The third part (for $x\geq 1$) has equation $y=-x+2$. This gives
$$\mathrm{f}(x)=\begin{cases}4 & \text{for } x<-2\\ x^2 & \text{for } -2\leq x<1\\ -x+2 & \text{for } x\geq 1\end{cases}$$

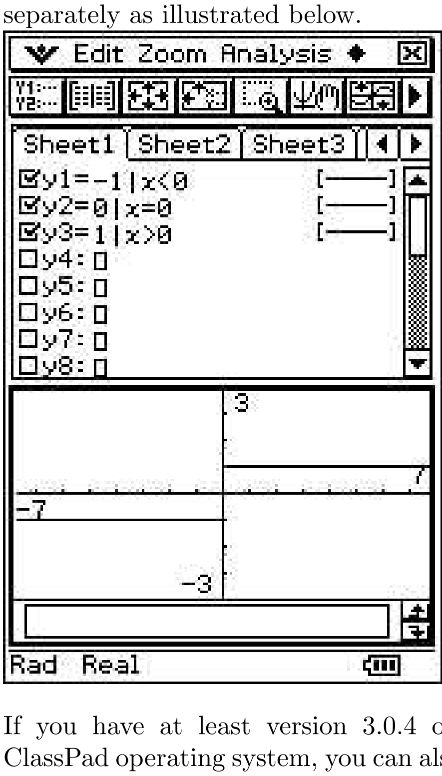
8. Working left to right, the first part (for $x\leq -2$) has equation $y=2$. The second part (for $-2<x<2$) has equation $y=x$. The third part (for $x=2$) has equation $y=1$. The fourth part (for $x>2$) has equation $y=0.5x-1$. This gives
$$\mathrm{f}(x)=\begin{cases}2 & \text{for } x\leq -2\\ x & \text{for } -2<x<2\\ 1 & \text{for } x=2\\ 0.5x-1 & \text{for } x>2\end{cases}$$

9. Refer to solutions in Sadler.

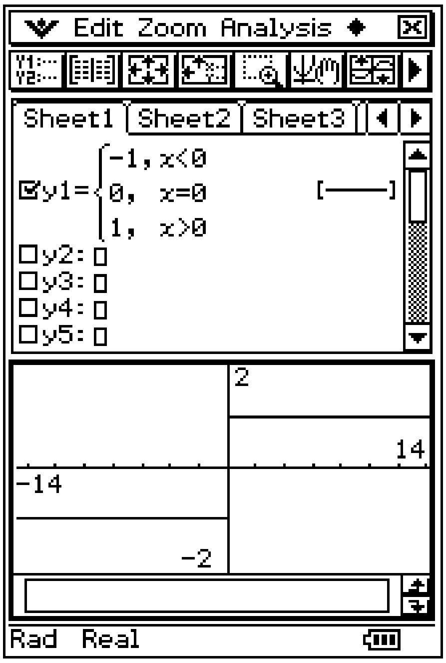
10. Refer to solutions in Sadler.

11. (a)
$$\mathrm{sgn}(x)=\begin{cases}-1 & \text{for } x<0\\ 0 & \text{for } x=0\\ 1 & \text{for } x>0\end{cases}$$
   (b) Refer to solutions in Sadler.
   (c) This can be done several ways. The classpad has a "signum" function that is the same as $\mathrm{sgn}(x)$. It may we worth showing here how to graph this in a way that reflects the piecewise definition.

   Graphing a piecewise-defined function on the ClassPad involves entering the pieces separately as illustrated below.



   If you have at least version 3.0.4 of the ClassPad operating system, you can also enter the piecewise-defined function directly:



12. (a) See the solution in Sadler.

 (b) The ClassPad has an int function.

 (c)

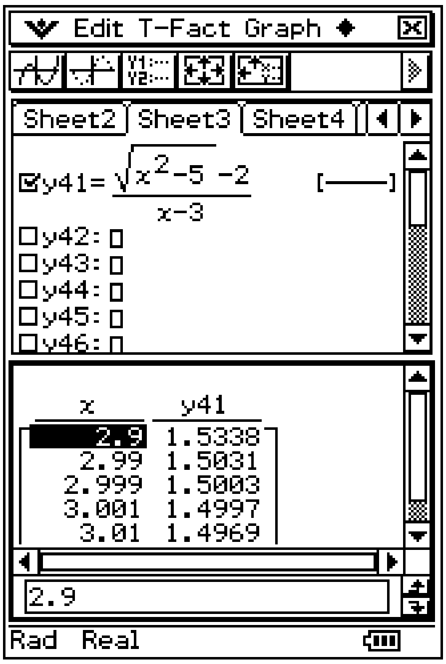
$$\text{int}(x) = \begin{cases} -1 & \text{for } -1 \le x < 0 \\ 0 & \text{for } 0 \le x < 1 \\ 1 & \text{for } 1 \le x < 2 \\ 2 & \text{for } x = 2 \end{cases}$$

In fact with a bit of ingenuity we can also come up with a recursive definition that works for a domain of $x \in \Re$:

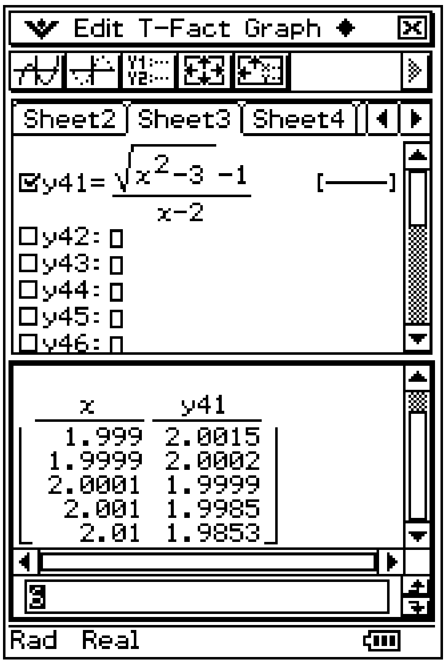
$$\text{int}(x) = \begin{cases} \text{int}(x+1) - 1 & \text{for } x < 0 \\ 0 & \text{for } 0 \le x < 1 \\ \text{int}(x-1) + 1 & \text{for } x \ge 0 \end{cases}$$

Try to understand how this works by using the definition to find, for instance, int(2.3).

Exercise 6C

1. 

The limit appears to be 1.5.

2. 

The limit appears to be 2.

3. f(x) is continuous at a so
$$\lim_{x\to a} \mathrm{f}(x) = f(a) = 10$$

4. $\lim_{x\to a^-} \mathrm{f}(x) = \lim_{x\to a^+} \mathrm{f}(x) = 10$ so
$$\lim_{x\to a} \mathrm{f}(x) = 10$$
(The actual value of f(a) is of no relevance.)

5. $\lim_{x\to a^-} \mathrm{f}(x) = 10$; $\lim_{x\to a^+} \mathrm{f}(x) = 5$ The limits from above and below are not equal so $\lim_{x\to a} \mathrm{f}(x)$ does not exist.

6. $\lim_{x\to a^-} \mathrm{f}(x) = 10$; $\lim_{x\to a^+} \mathrm{f}(x) = 0$. The limits from above and below are not equal so $\lim_{x\to a} \mathrm{f}(x)$ does not exist. (The actual value of f(a) is of no relevance.)

7. f(x) is continuous at a so
$$\lim_{x\to a} \mathrm{f}(x) = \mathrm{f}(a) = 10$$

8. f(x) is continuous at a so
$$\lim_{x\to a} \mathrm{f}(x) = \mathrm{f}(a) = 5$$

9. $\lim_{x\to a^-} \mathrm{f}(x) = 5$; $\lim_{x\to a^+} \mathrm{f}(x) = 10$. The limits from above and below are not equal so $\lim_{x\to a} \mathrm{f}(x)$ does not exist.

10. $\lim_{x\to a^-} \mathrm{f}(x) = \lim_{x\to a^+} \mathrm{f}(x) = 10$ so
$$\lim_{x\to a} \mathrm{f}(x) = 10$$
(It makes no difference that f(x) is not defined at $x = a$.)

11. $\lim_{x\to a^-} \mathrm{f}(x) = \lim_{x\to a^+} \mathrm{f}(x) = 10$ so
$$\lim_{x\to a} \mathrm{f}(x) = 10$$
(The discontinuity does not matter here because we only care about what happens in the vicinity of $x = a$.)

12. $\lim_{x\to a^-} \mathrm{f}(x) = -\infty$; $\lim_{x\to a^+} \mathrm{f}(x) = +\infty$. The limits from above and below do not exist (differently) so $\lim_{x\to a} \mathrm{f}(x)$ does not exist.

13. $\lim_{x\to a^-} \mathrm{f}(x) = \lim_{x\to a^+} \mathrm{f}(x) = 2$ so
$$\lim_{x\to a} \mathrm{f}(x) = 2$$
(The discontinuity does not matter here because we only care about what happens in the vicinity of $x = a$.)

14. $\lim_{x\to a^-} \mathrm{f}(x) = +\infty$; $\lim_{x\to a^+} \mathrm{f}(x) = +\infty$. The limits from above and below do not exist so $\lim_{x\to a} \mathrm{f}(x)$ does not exist.

15. $3x + 5$ is continuous everywhere, so
$$\lim_{x\to 1}(3x+5) = 3\times 1 + 5 = 8$$

16. $2x^2 + x + 3$ is continuous everywhere, so
$$\lim_{x\to 1}(2x^2 + x + 1) = 2(1)^2 + 1 + 3 = 6$$

17. $\frac{5}{x-2}$ has a discontinuity at $x = 2$ but is continuous at $x = 4$ so
$$\lim_{x\to 4} \frac{5}{x-2} = \frac{5}{4-2} = 2.5$$

18. $\frac{x+3}{x+2}$ has a discontinuity at $x = -2$ but is continuous at $x = 2$ so
$$\lim_{x\to 4} \frac{x+3}{x+2} = \frac{2+3}{2+2} = 1.25$$

19. $\frac{4x^2-36}{x-3}$ is not defined at $x = 3$ and gives $\frac{0}{0}$ so it needs further investigation. We can factorise and simplify to obtain a function that is identical except for being defined at $x = 3$ thus:
$$\begin{aligned}\frac{4x^2-36}{x-3} &= \frac{4(x^2-9)}{x-3}\\ &= \frac{4(x+3)(x-3)}{x-3}\\ &= 4(x+3) \qquad (x \neq 3)\\ \therefore \quad \lim_{x\to 3}\frac{4x^2-36}{x-3} &= 4(3+3)\\ &= 24\end{aligned}$$

20. $\frac{x^2+3x-10}{x-2}$ is not defined at $x = 2$ and gives $\frac{0}{0}$ so it needs further investigation. We can factorise and simplify to obtain a function that is identical except for being defined at $x = 2$ thus:
$$\begin{aligned}\frac{x^2+3x-10}{x-2} &= \frac{(x+5)(x-2)}{x-2}\\ &= (x+5) \qquad (x \neq 2)\\ \therefore \quad \lim_{x\to 2}\frac{x^2+3x-10}{x-2} &= 2+5\\ &= 7\end{aligned}$$

21. $\frac{x^2-3x}{x-3}$ is not defined at $x = 3$ and gives $\frac{0}{0}$ so it needs further investigation. We can factorise and simplify to obtain a function that is identical except for being defined at $x = 3$ thus:
$$\begin{aligned}\frac{x^2-3x}{x-3} &= \frac{x(x-3)}{x-3}\\ &= x \qquad (x \neq 3)\\ \therefore \quad \lim_{x\to 3}\frac{x^2-3x}{x-3} &= 3\end{aligned}$$

22. $\frac{2x^2-10x}{x-5}$ is not defined at $x = 5$ and gives $\frac{0}{0}$ so it needs further investigation. We can factorise and simplify to obtain a function that is identical except for being defined at $x = 5$ thus:

$$\begin{aligned} \frac{2x^2-10x}{x-5} &= \frac{2x(x-5)}{x-5} \\ &= 2x \qquad (x \neq 5) \\ \therefore \quad \lim_{x\to 5} \frac{2x^2-10x}{x-5} &= 2 \times 5 \\ &= 10 \end{aligned}$$

23. $\frac{x^2+5x+6}{x-1}$ is continuous at $x = 2$ (its only discontinuity is at $x = 1$) so

$$\begin{aligned} \lim_{x\to 2} \frac{x^2+5x+6}{x-1} &= \frac{2^2+5\times 2+6}{2-1} \\ &= 20 \end{aligned}$$

24. $\frac{3x^2-12}{x-2}$ is not defined at $x = 2$ and gives $\frac{0}{0}$ so it needs further investigation. We can factorise and simplify to obtain a function that is identical except for being defined at $x = 2$ thus:

$$\begin{aligned} \frac{3x^2-12}{x-2} &= \frac{3(x^2-4)}{x-2} \\ &= \frac{3(x+2)(x-2)}{x-2} \\ &= 3(x+2) \qquad (x \neq 2) \\ \therefore \quad \lim_{x\to 2} \frac{3x^2-12}{x-2} &= 3(2+2) \\ &= 12 \end{aligned}$$

25. $(x+2)^3$ is a polynomial function so it is continuous everywhere, so

$$\begin{aligned} \lim_{x\to 2}(x+2)^3 &= (2+2)^3 \\ &= 64 \end{aligned}$$

26. If we simply substitute $x = 5$ we get

$$\begin{aligned} \frac{x^2-4x-5}{x^2-7x+10} &= \frac{25-20-5}{25-35+10} \\ &= \frac{0}{0} \end{aligned}$$

so we need further investigation. Factorising and simplifying gives:

$$\begin{aligned} \frac{x^2-4x-5}{x^2-7x+10} &= \frac{(x-5)(x+1)}{(x-5)(x-2)} \\ &= \frac{x+1}{x-2} \qquad (x \neq 5) \\ \therefore \quad \lim x \to 5 \frac{x^2-4x-5}{x^2-7x+10} &= \frac{5+1}{5-2} \\ &= 2 \end{aligned}$$

27. If we simply substitute $x = 1$ we get

$$\begin{aligned} \frac{x^2-1}{x^2-x} &= \frac{1-1}{1-1} \\ &= \frac{0}{0} \end{aligned}$$

so we need further investigation. Factorising and simplifying gives:

$$\begin{aligned} \frac{x^2-1}{x^2-x} &= \frac{(x+1)(x-1)}{x(x-1)} \\ &= \frac{x+1}{x} \qquad (x \neq 1) \\ \therefore \quad \lim x \to 1 \frac{x^2-1}{x^2-x} &= \frac{1+1}{1} \\ &= 2 \end{aligned}$$

28. If we simply substitute $x = 5$ we get

$$\frac{2+x}{5-x} = \frac{7}{0}$$

The limit does not exist: the denominator approaches zero as the numerator approaches 7 so the limit increases or decreases without bound, depending on whether we approach from above or below. ($\lim_{x\to 5^-} 2+x5-x = +\infty$ since the denominator is positive for $x < 5$ and $\lim_{x\to 5^+} 2+x5-x = -\infty$ since the denominator is negative for $x > 5$. The numerator is positive in either case.)

29. If we simply substitute $x = 3$ we get

$$\frac{x-5}{x-3} = \frac{-2}{0}$$

The limit does not exist: the denominator approaches zero as the numerator approaches -2 so the limit increases or decreases without bound, depending on whether we approach from above or below. ($\lim_{x\to 3^-} x-5x-3 = +\infty$ since the denominator is negative for $x < 3$ and $\lim_{x\to 5^+} x-5x-3 = -\infty$ since the denominator is positive for $x > 3$. The numerator is negative in either case.)

30. If we simply substitute $x = 0$ we get

$$\frac{x^2+4x}{x} = \frac{0}{0}$$

so we need further investigation. Factorising and simplifying gives:

$$\begin{aligned} \frac{x^2+4x}{x} &= \frac{x(x+4)}{x} \\ &= x+4 \qquad (x \neq 0) \\ \therefore \quad \lim x \to 0 \frac{x^2+4x}{x} &= 0+4 \\ &= 4 \end{aligned}$$

31. If we simply substitute $x = 1$ we get
$$\frac{5x^2-5}{2x-2} = \frac{5-5}{2-2} = \frac{0}{0}$$
so we need further investigation. Factorising and simplifying gives:
$$\begin{aligned}\frac{5x^2-5}{2x-2} &= \frac{5(x^2-1)}{2(x-1)}\\ &= \frac{5(x+1)(x-1)}{2(x-1)}\\ &= \frac{5(x+1)}{2} \qquad (x \neq 1)\\ \therefore \quad \lim x \to 1 \frac{5x^2-5}{2x-2} &= \frac{5(1+1)}{2}\\ &= 5\end{aligned}$$

32. If we simply substitute $x = 2$ we get
$$\frac{3x-6}{x-2} = \frac{6-6}{2-2} = \frac{0}{0}$$
so we need further investigation. Factorising and simplifying gives:
$$\begin{aligned}\frac{3x-6}{x-2} &= \frac{3(x-2)}{x-2}\\ &= 3 \qquad (x \neq 1)\\ \therefore \quad \lim x \to 2 \frac{3x-6}{x-2} &= 3\end{aligned}$$

33.
$$\begin{aligned}\lim_{x\to4^-} f(x) &= 2(4) - 3\\ &= 5\\ \lim_{x\to4^+} f(x) &= (4) + 1 \qquad = 5\\ \therefore \quad \lim_{x\to4} f(x) &= 5\end{aligned}$$

34.
$$\begin{aligned}\lim_{x\to2^-} f(x) &= 3(2) - 2\\ &= 4\\ \lim_{x\to2^+} f(x) &= 2(2) - 3\\ &= 1\\ \therefore \quad \lim_{x\to2} f(x) &\text{ does not exist.}\end{aligned}$$

35. $f(x)$ is continuous at $x = 3$ so $\lim_{x\to3} f(x) = f(3) = 2(3) - 4 = 2$.

36.
$$\begin{aligned}\lim_{x\to5^-} f(x) &= 3(5) - 2\\ &= 13\\ \lim_{x\to5^+} f(x) &= 13(5)\\ &= 65\\ \therefore \quad \lim_{x\to5} &\text{ does not exist.}\end{aligned}$$

37.
$$\begin{aligned}\lim_{x\to3^-} f(x) &= (3)^2\\ &= 9\\ \lim_{x\to3^+} f(x) &= 3(3)\\ &= 9\\ \therefore \quad \lim_{x\to3} &= 9\end{aligned}$$

38.
$$\begin{aligned}\lim_{x\to2^-} f(x) &= (2-1)^2\\ &= 1\\ \lim_{x\to2^+} f(x) &= 3(2) - 5\\ &= 1\\ \therefore \quad \lim_{x\to2} &= 1\end{aligned}$$

39. (a) As $x \to 0^+$, $x > 0$ so $|x| = x$ and $\frac{|x|}{x} = 1$ so $\lim_{x\to0^+} \frac{|x|}{x} = 1$.

(b) As $x \to 0^-$, $x < 0$ so $|x| = -x$ and $\frac{|x|}{x} = -1$ so $\lim_{x\to0^-} \frac{|x|}{x} = -1$.

(c) As $\lim_{x\to0^+} \frac{|x|}{x} \neq \lim_{x\to0^-} \frac{|x|}{x}$ we conclude that $\lim_{x\to0} \frac{|x|}{x}$ does not exist.

40. (a) For $x > 0$ there is no discontinuity in $\sqrt{x}$ so $\lim_{x\to0^+} \sqrt{x} = \sqrt{0} = 0$.

(b) For $x < 0$, $\sqrt{x}$ has no real value, so $\lim_{x\to0^-} \sqrt{x}$ does not exist.

41. (a) $\lim_{x\to1^+} \text{sgn}x = 1$ (since $\text{sgn}x = 1 \quad \forall x > 0$.)

(b) $\lim_{x\to1^-} \text{sgn}x = 1$ (since $\text{sgn}x = 1 \quad \forall x > 0$.)

(c) $\lim_{x\to1} \text{sgn}x = 1$ (since $\text{sgn}x = 1 \quad \forall x > 0$.)

(d) $\lim_{x\to-1^+} \text{sgn}x = -1$ (since $\text{sgn}x = -1 \quad \forall x < 0$.)

(e) $\lim_{x\to-1^-} \text{sgn}x = -1$ (since $\text{sgn}x = -1 \quad \forall x < 0$.)

(f) $\lim_{x\to-1} \text{sgn}x = -1$ (since $\text{sgn}x = -1 \quad \forall x < 0$.)

(g) $\lim_{x\to0^+} \text{sgn}x = 1$ (since $\text{sgn}x = 1 \quad \forall x > 0$.)

(h) $\lim_{x\to0^-} \text{sgn}x = -1$ (since $\text{sgn}x = -1 \quad \forall x < 0$.)

(i) $\lim_{x\to0} \text{sgn}x$ does not exist (because $\lim_{x\to0^+} \text{sgn}x \neq \lim_{x\to0^-} \text{sgn}x$).

Exercise 6D

1. Considering dominant powers as $x \to \infty$,

$$\frac{3x^2+2x-1}{x} = \frac{3x^2}{x} = 3x$$

so

$$\lim_{x\to\infty} \frac{3x^2+2x-1}{x} = \infty$$

that is, $\lim_{x\to\infty} \frac{3x^2+2x-1}{x}$ does not exist.

2. Considering dominant powers as $x \to \infty$,

$$\frac{3x^2+2x-1}{x^2} = \frac{3x^2}{x^2} = 3$$

so

$$\lim_{x\to\infty} \frac{3x^2+2x-1}{x^2} = 3$$

3. Considering dominant powers as $x \to \infty$,

$$\frac{3x^2+2x-1}{x^3} = \frac{3x^2}{x^3} = \frac{3}{x}$$

so

$$\lim_{x\to\infty} \frac{3x^2+2x-1}{x^3} = 0$$

4. Considering dominant powers as $x \to \infty$,

$$\frac{2x^2-x}{5-x^2} = \frac{2x^2}{-x^2} = -2$$

so

$$\lim_{x\to\infty} \frac{2x^2-x}{5-x^2} = -2$$

5. Considering dominant powers as $x \to \infty$,

$$\frac{4x^3+2x-3}{5x^3+2x^2} = \frac{4x^3}{5x^3} = 0.8$$

so

$$\lim_{x\to\infty} \frac{4x^3+2x-3}{5x^3+2x^2} = 0.8$$

6. Considering dominant powers as $x \to -\infty$,

$$\frac{3x^4}{2x^3+x-2} = \frac{3x^4}{2x^3} = 1.5x$$

so

$$\lim_{x\to-\infty} \frac{3x^4}{2x^3+x-2} = -\infty$$

that is, $\lim_{x\to\infty} \frac{3x^4}{2x^3+x-2}$ does not exist.

7. Considering dominant powers as $x \to -\infty$,

$$\frac{3x^2}{2x^3+x-2} = \frac{3x^2}{2x^3} = \frac{1.5}{x}$$

so

$$\lim_{x\to-\infty} \frac{3x^2}{2x^3+x-2} = 0$$

8. Considering dominant powers as $x \to -\infty$,

$$\frac{3x^3}{2x^3+x-2} = \frac{3x^3}{2x^3} = 1.5$$

so

$$\lim_{x\to-\infty} \frac{3x^2}{2x^3+x-2} = 1.5$$

9. First simplify the algebraic fraction then consider dominant powers as $x \to \infty$:

$$\frac{(2x+3)(2x-5)}{(3-5x)(2x-5)} = \frac{2x+3}{3-5x} = -0.4$$

so

$$\lim_{x\to\infty} \frac{(2x+3)(2x-5)}{(3-5x)(2x-5)} = -0.4$$

10. Considering dominant powers as $x \to \infty$, the numerator tends to $7x^2$ and the denominator to $-4x^2$ so

$$\frac{7x^2+x}{(3-4x)(1+x)} = \frac{7x^2}{-4x^2} = -1.75$$

so

$$\lim_{x\to\infty} \frac{7x^2+x}{(3-4x)(1+x)} = -1.75$$

11. (a)
$$\begin{aligned}\lim_{x\to2^-} \mathrm{f}(x) &= \lim_{x\to2^-} x^2\\ &= 4\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x\to2^+} \mathrm{f}(x) &= \lim_{x\to2^+} \frac{3x-2}{x-1}\\ &= \frac{3(2)-2}{2-1}\\ &= 4\end{aligned}$$

(c) $\lim_{x\to2} \mathrm{f}(x) = 4$

(d)
$$\begin{aligned}\lim_{x\to-\infty} \mathrm{f}(x) &= \lim_{x\to-\infty} x^2\\ &= \infty\end{aligned}$$
(i.e. the limit does not exist.)

(e)
$$\begin{aligned}\lim_{x\to\infty} \mathrm{f}(x) &= \lim_{x\to\infty} \frac{3x-2}{x-1}\\ &= 3\end{aligned}$$

Exercise 6E

1. $f(x)$ is discontinuous where the denominator is equal to zero, i.e.

$$\begin{aligned} x^2 - 3x - 10 &= 0 \\ (x-5)(x+2) &= 0 \\ x &= 5 \\ \text{or}\quad x &= -2 \end{aligned}$$

2. $f(x)$ is discontinuous where the denominator is equal to zero, i.e.

$$\begin{aligned} x(2x-1)(x-1) &= 0 \\ x &= 0 \\ \text{or}\quad 2x-1 &= 0 \\ x &= 0.5 \\ \text{or}\quad x-1 &= 0 \\ x &= 1 \end{aligned}$$

3. Each piece is a polynomial function and hence continuous and the function is defined $\forall x \in \Re$ so the only possible discontinuities are where the pieces join.

$$\begin{aligned} \lim_{x\to -2^-} f(x) &= a(-2) + 10 \\ &= 10 - 2a \\ \lim_{x\to -2^+} f(x) &= 2 - (-2) \\ &= 4 \end{aligned}$$

for the function to be continuous,

$$\begin{aligned} \lim_{x\to -2^-} f(x) &= \lim_{x\to -2^+} f(x) \\ 10 - 2a &= 4 \\ -2a &= -6 \\ a &= 3 \\ \lim_{x\to 0^-} f(x) &= 2 - (0) \\ &= 2 \\ \lim_{x\to 0^+} f(x) &= 3(0) + b \\ &= b \end{aligned}$$

for the function to be continuous,

$$\begin{aligned} \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^-} f(x) \\ b &= 2 \end{aligned}$$

4. The function must be uniquely defined $\forall x \in \Re$ so this gives us a value for c: $c = 4$. Each piece is a polynomial function and hence continuous so the only other possible discontinuities are where the pieces join.

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= 5(0) + a \\ &= a \\ \lim_{x\to 0^+} f(x) &= b(0) - 3 \\ &= -3 \end{aligned}$$

for the function to be continuous,

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^+} f(x) \\ a &= -3 \\ \lim_{x\to 4^-} f(x) &= b(4) - 3 \\ &= 4b - 3 \\ \lim_{x\to 4^+} f(x) &= (4) + 1 \\ &= 5 \end{aligned}$$

for the function to be continuous,

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^+} f(x) \\ 4b - 3 &= 5 \\ b &= 2 \end{aligned}$$

5. Each individual part is a polynomial function and the function is defined $\forall x \in \Re$ so the only possible discontinuities are where the parts join.

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= 1^2 \\ &= 1 \\ \lim_{x\to 1^+} f(x) &= 3(1) - 2 \\ &= 1 \\ f(1) &= 3(1) - 2 \\ &= 1 \end{aligned}$$

so the function is continuous at $x = 1$.

$$\begin{aligned} \lim_{x\to 3^-} f(x) &= 3(3) - 2 \\ &= 7 \\ \lim_{x\to 3^+} f(x) &= (3)^2 + 2 \\ &= 11 \end{aligned}$$

The function has a single discontinuity at $x = 3$ and is continuous everywhere else.

6.
   - The first piece is a polynomial function, so $f(x)$ is continuous for $x < 4$.
   - $$\begin{aligned} \lim_{x\to 4^-} f(x) &= (4) + 1 \\ &= 5 \\ \lim_{x\to 4^+} f(x) &= \sqrt{(4)} + 3 \\ &= 5 \\ f(4) &= \sqrt{(4)} + 3 \\ &= 5 \end{aligned}$$

     $f(x)$ is continuous at $x = 4$.

- The second piece is defined and continuous for all non-negative x so f(x) is continuous for $4 < x < 9$.
- $$\begin{aligned}\lim_{x\to 9^-} \mathrm{f}(x) &= \sqrt{(9)} + 3\\ &= 6\\ \lim_{x\to 9^+} \mathrm{f}(x) &= \frac{18}{12-(9)}\\ &= 6 \mathrm{f}(9) \qquad = \sqrt{(9)} + 3\\ &= 6\end{aligned}$$

  f(x) is continuous at $x = 9$.
- The third piece is continuous for all $x \neq 12$. The function is undefined at $x = 12$ so there is a discontinuity at $x = 12$.
- There is a discontinuity at $x = 30$ because f(30) is undefined.
- The fourth piece is a polynomial function and hence continuous everywhere, so the function is continuous for $x > 30$.

Exercise 6F

1. Not differentiable: f(a) is undefined.
2. Not differentiable: the gradient function from the left is not equal to that from the right (it is not smooth at $x = a$).
3. Not differentiable: the gradient function from the left is not equal to that from the right (it is not smooth at $x = a$).
4. Not differentiable: f(a) is undefined.
5. Not differentiable: the function is not continuous at $x = a$.
6. Not differentiable: f(a) is undefined.
7. Differentiable: f(x) is continuous and smooth at $x = a$.
8. Not differentiable: the function is not continuous at $x = a$.
9. Differentiable: f(x) is continuous and smooth at $x = a$.
10. Differentiable. (Polynomial functions are differentiable everywhere.)
11. Differentiable. (Polynomial functions are differentiable everywhere.)
12. Not differentiable: the function is not continuous at $x = 1$.
13. Differentiable: the function is smooth and continuous at $x = -1$.
14. Differentiable: the function is smooth and continuous at $x = 3$.
15. Not differentiable: the derivative from the left is -2 and the derivative from the right is 2 so the function is not smooth at $x = 2.5$.
16. f(x) is continuous (it approaches 1 from left and right as $x \to 1$) so we must consider the derivative from the left and the right.

    From the left (i.e. for $x < 1$) we obtain $f'(x) = 3x^2$ and as $x \to 1^-$, $\mathrm{f}'(x) \to 3(1)^2 = 3$.

    From the right (i.e. for $x > 1$) we obtain $f'(x) = 1$ and as $x \to 1^+$, $\mathrm{f}'(x) \to 1$.

    $\therefore$ f(x) is not differentiable at $x = 1$.
17. f(x) is not continuous at $x = 1$ because

    $$\lim_{x\to 1^-} = (1)^3 = 1$$

    but

    $$\lim_{x\to 1^+} = 3(1) = 3$$

    $\therefore$ f(x) is not differentiable at $x = 1$.
18. f(x) is continuous at $x = 3$ (it approaches 9 from left and right as $x \to 3$) so we must consider the derivative from the left and the right.

    From the left (i.e. for $x < 3$) we obtain $f'(x) = 6$.

    From the right (i.e. for $x > 3$) we obtain $f'(x) = 2x$ and as $x \to 3^+$, $\mathrm{f}'(x) \to 6$.

    $\therefore$ f(x) is differentiable at $x = 3$.
19. f(x) is not defined at $x = 3$ and hence is not continuous and therefore not differentiable.
20.
    - For $x < 0$, f(x) is a polynomial function and hence both continuous and differentiable.
    - At $x = 0$ the function is continuous (it is defined and the limit approaches 0 from both directions). It is not, however, differentiable, because from the left is $\mathrm{f}'(x) = 6$ and from the right $\mathrm{f}'(x) = 6x$ so as $x \to 0$, $\mathrm{f}'(x) \to 0$.

- For $0 < x < 1$, $f(x)$ is a polynomial function and hence both continuous and differentiable.
- At $x = 1$ the function is not continuous as the limit from the left is $3(1)^2 = 3$ and from the right $2(1)^3 = 2$. Since it is not continuous it can not be differentiable.
- For $x > 1$, $f(x)$ is a polynomial function and hence both continuous and differentiable.

21.
- For $x < 1$, $f(x)$ is both continuous and differentiable.
- At $x = 1$, $f(x)$ is continuous but not differentiable since the gradient from the left is -1 and the gradient from the right is 1.
- For $1 < x < 4$, $f(x)$ is both continuous and differentiable.
- At $x = 4$, $f(x)$ is continuous ($\lim_{x\to 4^-} f(x) = |(4) - 1| = 3$ and $\lim_{x\to 4^+} f(x) = (4)^2 - 7(4) + 15 = 3$). From the left the derivative is $f'(x) = 1$; from the right $f'(x) = 2x - 7$ so as $x \to 4^+$, $f'(x) = 1$. Thus $f(x)$ is differentiable at $x = 4$.
- For $4 < x < 5$, $f(x)$ is both continuous and differentiable.
- At $x = 5$, $f(x)$ is not continuous ($\lim_{x\to 5^-} f(x) = (5)^2 - 7(5) + 15 = 5$ and $\lim_{x\to 5^+} f(x) = 3(5) = 15$) and therefore not differentiable.
- For $x > 5$, $f(x)$ is both continuous and differentiable.

22.
- For continuity at $x = 1$,

$$\begin{aligned}\lim_{x\to 1^-} f(x) &= \lim_{x\to 1^+} f(x)\\ 5(1) + 7 &= a(1)^2 + b(1) + c\\ a + b + c &= 12 \qquad (1)\end{aligned}$$

- For differentiability at $x = 1$

$$\begin{aligned}f'(x) \text{ from the left} &= f'(x) \text{ from the right}\\ 5 &= 2ax + b\end{aligned}$$

and as $x \to 1$ we get

$$2a + b = 5 \qquad (2)$$

- For continuity at $x = 3$,

$$\begin{aligned}\lim_{x\to 3^-} f(x) &= \lim_{x\to 3^+} f(x)\\ a(3)^2 + b(3) + c &= d \qquad (3)\end{aligned}$$

- For differentiability at $x = 3$

$$\begin{aligned}f'(x) \text{ from the left} &= f'(x) \text{ from the right}\\ 2ax + b &= 0\end{aligned}$$

and as $x \to 3$ we get

$$6a + b = 0 \qquad (4)$$

Now solving these four equations simultaneously, from (2) and (4) we get

$$\begin{aligned}4a &= -5\\ a &= -1.25\\ 6a + b &= 0\\ -7.5 + b &= 0\\ b &= 7.5\end{aligned}$$

Now substitute into (1)

$$\begin{aligned}a + b + c &= 12\\ -1.25 + 7.5 + c &= 12\\ c &= 5.75\end{aligned}$$

Finally equation (3) to find d:

$$\begin{aligned}9a + 3b + c &= d\\ d &= 9(-1.25) + 3(7.5) + 5.75\\ &= 17\end{aligned}$$

23.
- For continuity at $x = -1$,

$$\begin{aligned}\lim_{x\to -1^-} f(x) &= \lim_{x\to -1^+} f(x)\\ a(-1)^3 &= 6(-1) + b\\ -a &= -6 + b \qquad (1)\end{aligned}$$

- For differentiability at $x = -1$

$$\begin{aligned}f'(x) \text{ from the left} &= f'(x) \text{ from the right}\\ 3ax^2 &= 6\end{aligned}$$

and as $x \to -1$ we get

$$\begin{aligned}3a(-1)^2 &= 6\\ 3a &= 6\\ a &= 2\end{aligned}$$

substituting into (1)

$$\begin{aligned}-2 &= -6 + b\\ b &= 4\end{aligned}$$

- For continuity at $x = 5$,

$$\begin{aligned}\lim_{x\to 5^-} f(x) &= \lim_{x\to 5^+} f(x)\\ 6(5) + b &= c(5)^2 + d(5) + 29\\ 30 + 4 &= 25c + 5d + 29\\ 25c + 5d &= 5\\ 5c + d &= 1 \qquad (2)\end{aligned}$$

- For differentiability at $x = 5$

$$\begin{aligned} f'(x) \text{ from the left} &= f'(x) \text{ from the right} \\ 6 &= 2cx + d \end{aligned}$$

and as $x \to 5$ we get

$$10c + d = 6 \qquad (3)$$

Now solving (2) and (3) simultaneously we get

$$\begin{aligned} 5c &= 5 \\ c &= 1 \\ 5c + d &= 1 \\ 5 + d &= 1 \\ d &= -4 \end{aligned}$$

Miscellaneous Exercise 6

1. (a) The amplitude is 10cm (half the peak-peak distance).

   (b) The period of the motion is 2 seconds (time taken for the cycle to repeat).

   (c) The weight passes through the equilibrium position twice per full cycle, so it passes 10 times in the first 10 seconds.

2. Left hand side:

$$\begin{aligned} \cos^4\theta - \sin^4\theta &= (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) \\ &= (1)(\cos 2\theta) \\ &= \cos 2\theta \end{aligned}$$

$\square$

3.

$$\begin{aligned} 2\cos^2 x + \sin x &= 2\cos 2x \\ 2(1 - \sin^2 x) + \sin x &= 2(1 - 2\sin^2 x) \\ 2 - 2\sin^2 x + \sin x &= 2 - 4\sin^2 x \\ 2\sin^2 x + \sin x &= 0 \\ \sin x(2\sin x + 1) &= 0 \end{aligned}$$

Using the null factor law:

$$\begin{aligned} \sin x &= 0 \\ x &= 0 \\ \text{or} \quad x &= \pi \\ \text{or} \quad x &= 2\pi \\ \text{or} \quad 2\sin x + 1 &= 0 \\ \sin x &= -\frac{1}{2} \\ x &= \frac{7\pi}{6} \\ \text{or} \quad x &= \frac{11\pi}{6} \end{aligned}$$

4. (a)

$$\begin{aligned} \sqrt{2^2 + 5^2} &= \sqrt{29} \\ \cos\alpha &= \frac{2}{\sqrt{29}} \\ \sin\alpha &= \frac{5}{\sqrt{29}} \\ \alpha &= \sin^{-1}\frac{5}{\sqrt{29}} \\ &= 68.2° \\ 2\cos\theta + 5\sin\theta &= \sqrt{29}\left(\frac{2}{\sqrt{29}}\cos\theta + \frac{5}{\sqrt{29}}\sin\theta\right) \\ &= \sqrt{29}(\cos\theta\cos\alpha + \sin\theta\sin\alpha) \\ &= \sqrt{29}\cos(\theta - \alpha) \\ &= \sqrt{29}\cos(\theta - 68.2°) \end{aligned}$$

   (b) The minimum value is $-\sqrt{29} \approx 5.4$.

$$\begin{aligned} \sqrt{29}\cos(\theta - 68.2°) &= -\sqrt{29} \\ \cos(\theta - 68.2°) &= -1 \\ \theta - 68.2° &= 180° \\ \theta &= 248.2° \end{aligned}$$

5. (a) Since z has no real component, $\bar{z} = -z = 3\sqrt{5}\mathrm{i}$

   (b) $$\begin{aligned} z^2 &= (-3\sqrt{5}\mathrm{i})^2 \\ &= 9 \times 5 \times -1 \\ &= -45 \end{aligned}$$

   (c) $1 - z^2 = 1 - -45 = 46$

   (d) $$\begin{aligned} (1 - z)^2 &= (1 - -3\sqrt{5}\mathrm{i})^2 \\ &= (1 + 3\sqrt{5}\mathrm{i})^2 \\ &= 1 + 6\sqrt{5}\mathrm{i} - 45 \\ &= -44 + 6\sqrt{5}\mathrm{i} \end{aligned}$$

6. If $z = a + b\mathrm{i}$ and $w = c + d\mathrm{i}$ then the product is

$$\begin{aligned} zw &= (a + b\mathrm{i})(c + d\mathrm{i}) \\ &= (ac - bd) + (ad + bc)\mathrm{i} \end{aligned}$$

The conjugate of the product is

$$\overline{zw} = (ac - bd) - (ad + bc)\mathrm{i}$$

The product of the conjugates is

$$\begin{aligned}\bar{z}\bar{w} &= (a - b\mathrm{i})(c - d\mathrm{i})\\ &= (ac - bd) + (-ad - bc)\mathrm{i}\\ &= (ac - bd) - (ad + bc)\mathrm{i}\end{aligned}$$

$\therefore$ The conjugate of the product is equal to the product of the conjugates.

7.
$$\begin{aligned}x^2 + 6x + y^2 - 10y &= 15\\ (x+3)^2 - 9 + (x-5)^2 - 25 &= 15\\ (x+3)^2 + (x-5)^2 &= 49\end{aligned}$$
The centre is at $(-3, 5)$ and the radius is 7.

8. (a) Expand the right hand side:

$$\begin{aligned}&(ax - b)(x^2 + cx + 4)\\ &= ax^3 + acx^2 + 4ax - bx^2 - bcx - 4b\\ &= ax^3 + (ac - b)x^2 + (4a - bc)x - 4b\end{aligned}$$

Now equate terms with corresponding powers of x gives

$$\begin{aligned}a &= 7\\ -4b &= -12\\ b &= 3\\ ac - b &= 4\\ 7c - 3 &= 4\\ 7c &= 7\\ c &= 1\end{aligned}$$

(b)
$$\begin{aligned}7x^3 + 4x^2 + 25x - 12 &= 0\\ (7x - 3)(x^2 + x + 4) &= 0\end{aligned}$$
The first factor gives

$$\begin{aligned}7x - 3 &= 0\\ x &= \frac{3}{7}\end{aligned}$$

The second factor gives

$$\begin{aligned}x^2 + x + 4 &= 0\\ x &= \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times 4}}{2 \times 1}\\ &= \frac{-1 \pm \sqrt{-15}}{2}\\ x &= -\frac{1}{2} + \frac{\sqrt{15}}{2}\mathrm{i}\\ \text{or}\quad x &= -\frac{1}{2} - \frac{\sqrt{15}}{2}\mathrm{i}\end{aligned}$$

9. For continuity at $x = 100$,

$$\begin{aligned}0.01(100)^2 - 1.2(100) + 50 &= a(100)^2 + b(100) - 250\\ 100 - 120 + 50 &= 10\,000a + 100b - 250\\ 280 &= 10\,000a + 100b\\ 100a + b &= 2.8 \qquad (1)\end{aligned}$$

The gradient at $x = 100$ gives:

$$\begin{aligned}0.02x - 1.2 &= 2ax + b\\ 0.02(100) - 1.2 &= 2a(100) + b\\ 2 - 1.2 &= 200a + b\\ 200a + b &= 0.8 \qquad (2)\end{aligned}$$

Solving (1) and (2) simultaneously

$$\begin{aligned}100a &= -2\\ a &= -0.02\\ 100a + b &= 2.8\\ -2 + b &= 2.8\\ b &= 4.8\end{aligned}$$

For continuity at $x = 150$

$$\begin{aligned}a(150)^2 + b(150) - 250 &= c(150)^2 + d(150) + 605\\ -0.02(22\,500) + 4.8(150) - 250 &= 22\,500c + 150d + 605\\ -450 + 720 - 250 &= 22\,500c + 150d + 605\\ 22\,500c + 150d &= 20 - 605\\ &= -585\\ 150c + d &= -3.9 \qquad (3)\end{aligned}$$

The gradient at $x = 150$ gives:

$$\begin{aligned}2ax + b &= 2cx + d\\ -0.04(150) + 4.8 &= 2c(150) + d\\ -6 + 4.8 &= 300c + d\\ 300c + d &= -1.2 \qquad (4)\end{aligned}$$

Solving (3) and (4) simultaneously

$$\begin{aligned}150c &= 2.7\\ c &= 0.018\\ 150c + d &= -3.9\\ 2.7 + d &= -3.9\\ d &= -6.6\end{aligned}$$

At point A, $x = 0$ and $y = e$ so

$$\begin{aligned}e &= 0.01(0)^2 - 1.2(0) + 50\\ &= 50\end{aligned}$$

At point B, $x = 100$ and $y = f$ so

$$\begin{aligned}f &= 0.01(100)^2 - 1.2(100) + 50\\ &= 100 - 120 + 50\\ &= 30\end{aligned}$$

At point C, $x = 150$ and $y = g$ so

$$\begin{aligned}g &= a(150)^2 + b(150) - 250\\ &= -0.02(22\,500) + 4.8(150) - 250\\ &= -450 + 720 - 250\\ &= 20\end{aligned}$$

At point D, $x = h$ and $y = 0$ so

$$\begin{aligned} ch^2 + dh + 605 &= 0 \\ 0.018h^2 - 6.6h + 605 &= 0 \\ h &= \frac{550}{3} \end{aligned}$$

(using either the quadratic formula or the calculator's solver for the last step.)

10.
$$\begin{aligned} \cos 3\theta &= \cos(2\theta + \theta) \\ &= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \\ &= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta \\ &= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta \\ &= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta \\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta \\ &= 4\cos^3\theta - 3\cos\theta \end{aligned}$$

Hence $a = 4$, $b = 0$, $c = -3$, $d = 0$.

11. First piece, $x < -2$: $\mathrm{f}(x)$ is a polynomial function and hence continuous and differentiable.

Where the first and second pieces meet, $x = -2$:

$$\begin{aligned} \lim_{x\to -2^-} \mathrm{f}(x) &= -6 - (-2)^2 \\ &= -10 \\ \lim_{x\to -2^+} \mathrm{f}(x) &= 4(-2) - 2 \\ &= -10 \end{aligned}$$

$\mathrm{f}(x)$ is continuous at $x = -2$.

Derivative from the left:

$$\begin{aligned} \mathrm{f}'(x) &= -2x \\ &= -2(-2) \\ &= 4 \end{aligned}$$

Derivative from the right:

$$\mathrm{f}'(x) = 4$$

$\mathrm{f}(x)$ is differentiable at $x = -2$.

Second piece, $-2 < x < 3$: $\mathrm{f}(x)$ is a polynomial function and hence continuous and differentiable.

At $x = 3$,

$$\begin{aligned} \lim_{x\to 3^-} \mathrm{f}(x) &= 4(3) - 2 \\ &= 10 \end{aligned}$$

However, $\mathrm{f}(3) = 5$ so since $\lim_{x\to 3} \mathrm{f}(x) \neq \mathrm{f}(3)$ we must conclude that $\mathrm{f}(x)$ is not continuous at $x = 3$ and therefore not differentiable either.

Last piece, $x > 3$: $\mathrm{f}(x)$ is a polynomial function and hence continuous and differentiable.

Conclusion: $\mathrm{f}(x)$ is both continuous and differentiable everywhere except at $x = 3$ where it is neither.

12. (a) This is trivial; simply substitute $x = a$ to get

$$\lim_{x\to a}(x^2 + 3x + 5) = a^2 + 3a + 5$$

(b) Here we can't just substitute because we get $\frac{0}{0}$ so we must try another approach:

$$\begin{aligned} \lim_{x\to a} \frac{2(x-a)}{x^2 - a^2} &= \lim_{x\to a} \frac{2(x-a)}{(x-a)(x+a)} \\ &= \lim_{x\to a} \frac{2}{(x+a)} \\ &= \frac{2}{2a} \\ &= \frac{1}{a} \end{aligned}$$

(c) Here again we can't just substitute because we get $\frac{0}{0}$ so we must try another approach:

$$\begin{aligned} \lim_{x\to\sqrt{a}} \frac{x - \sqrt{a}}{x^2 - a} &= \lim_{x\to\sqrt{a}} \frac{x - \sqrt{a}}{(x-\sqrt{a})(x+\sqrt{a})} \\ &= \lim_{x\to\sqrt{a}} \frac{1}{x + \sqrt{a}} \\ &= \frac{1}{2\sqrt{a}} \end{aligned}$$

13. First without a calculator:

$$\begin{aligned} \mathrm{f}'(x) &= \frac{2(2x+a) - 2(2x+3)}{(2x+a)^2} \\ &= \frac{2a-6}{(2x+a)^2} \end{aligned}$$

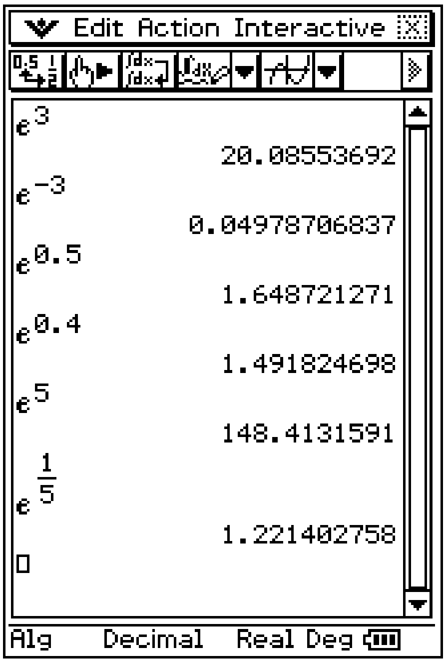
$$\begin{aligned} \mathrm{f}'(3) &= \frac{2a-6}{(6+a)^2} \\ \mathrm{f}'(3) &= -16 \\ \frac{2a-6}{(6+a)^2} &= -16 \\ 2a - 6 &= -16(6+a)^2 \\ 16(6+a)^2 + 2a - 6 &= 0 \\ 16(36 + 12a + a^2) + 2a - 6 &= 0 \\ 8(36 + 12a + a^2) + a - 3 &= 0 \\ 288 + 96a + 8a^2 + a - 3 &= 0 \\ 8a^2 + 97a + 285 &= 0 \end{aligned}$$

Use a calculator (or the quadratic formula) to solve this quadratic: $a = -5$ or $a = -7.125$.

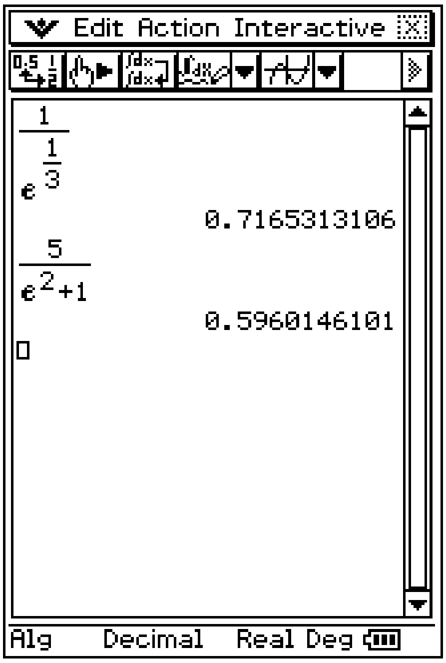
Chapter 7

Exercise 7A

1–6.



7–8.



9. (a) $t = 5 \times 60 = 300$

$$T = 10 + 65e^{-0.004\times300} = 29.6°\text{C}$$

(b) $t = 5 \times 60 = 600$

$$T = 10 + 65e^{-0.004\times600} = 15.9°\text{C}$$

10. $\log_e e = \log_e e^1 = 1$

11. $\log_e \frac{1}{e} = \log_e e^{-1} = -1$

12. $\log_e(e^3) = 3$

13. $\log_e \sqrt{e} = \log_e e^{\frac{1}{2}} = 0.5$

14. $$\begin{aligned} e^{x+1} &= 7 \\ x+1 &= \ln(7) \\ x &= \ln(7) - 1 \end{aligned}$$

15. $$\begin{aligned} e^{x+3} &= 50 \\ x+3 &= \ln(50) \\ x &= \ln(50) - 3 \end{aligned}$$

16. $$\begin{aligned} e^{x-3} &= 100 \\ x-3 &= \ln(100) \\ x &= \ln(100) + 3 \end{aligned}$$

17. $$\begin{aligned} e^{2x+1} &= 15 \\ 2x+1 &= \ln(15) \\ x &= \frac{\ln(15) - 1}{2} \end{aligned}$$

18. $$\begin{aligned} 5e^{3x-1} &= 3000 \\ e^{3x-1} &= 600 \\ 3x-1 &= \ln(600) \\ x &= \frac{\ln(600) + 1}{3} \end{aligned}$$

19. $$\begin{aligned} 4e^{x+2} + 3e^{x+2} &= 7\,000 \\ 7e^{x+2} &= 7\,000 \\ e^{x+2} &= 1\,000 \\ x+2 &= \ln(1\,000) \\ x &= \ln(1\,000) - 2 \end{aligned}$$

20. $$\begin{aligned} e^2x - 30e^x &= 200 \\ (e^x)^2 - 30e^x &= 200 \\ y^2 - 30y &= 200 = 0 \\ (y-10)(y-20) &= 0 \\ y &= 10 \qquad \text{or } y = 20 \\ e^x &= 10e^x \qquad = 20 \\ x &= \ln 10 \qquad x = \ln 20 \end{aligned}$$

21. $$\begin{aligned} A &= 2000e^{-t} \\ e^{-t} &= \frac{A}{2000} \\ e^t &= \frac{2000}{A} \\ t &= \ln\frac{2000}{A} \end{aligned}$$

(a) $t = \ln\frac{2000}{1500} = 0.288$

(b) $t = \ln\frac{2000}{500} = 1.386$

(c) $t = \ln\frac{2000}{50} = 3.689$

22. (a) In 2000, $t = 10$ so

$$\begin{aligned} P &= 20\,000\,000e^{0.02\times10} \\ &= 24\,428\,000 \end{aligned}$$

(b) In 2050, $t = 60$ so

$$\begin{aligned} P &= 20\,000\,000\mathrm{e}^{0.02\times 60} \\ &= 66\,402\,000 \end{aligned}$$

23. (a) $t = 0$ so

$$\begin{aligned} N &= 5\,000\mathrm{e}^{0.55\times 0} \\ &= 5\,000 \end{aligned}$$

(b) $t = 3$ so

$$\begin{aligned} N &= 5\,000\mathrm{e}^{0.55\times 3} \\ &= 26\,000 \end{aligned}$$

(c) $t = 10$ so

$$\begin{aligned} N &= 5\,000\mathrm{e}^{0.55\times 10} \\ &= 1\,233\,000 \end{aligned}$$

24. In 2010, $t = 20$ so the company's requirement is

$$P\mathrm{e}^{0.1\times 20} = 7.39P$$

The requirement has increased from 100% of P in 1990 to 739% of P in 2010, i.e. an increase of 639%.

25. (a) $$\begin{aligned} N &= \frac{100\,000}{1+499\mathrm{e}^{-0.8\times 0}} \\ &= 200 \end{aligned}$$

(b) $$\begin{aligned} N &= \frac{100\,000}{1+499\mathrm{e}^{-0.8\times 5}} \\ &= 9\,862 \end{aligned}$$

(c) $$\begin{aligned} N &= \frac{100\,000}{1+499\mathrm{e}^{-0.8\times 10}} \\ &= 85\,661 \end{aligned}$$

(d) $$\begin{aligned} \lim_{t\to\infty} \frac{100\,000}{1+499\mathrm{e}^{-0.8t}} &= \frac{100\,000}{1} \\ &= 100\,000 \end{aligned}$$

$$\left(\text{since } \lim_{t\to\infty} \mathrm{e}^{-0.08t} = \mathrm{e}^{-\infty} = \frac{1}{\mathrm{e}^{\infty}} = \frac{1}{\infty} = 0\right)$$

Exercise 7B

1. $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = e^x$

2. $\frac{\mathrm{d}}{\mathrm{d}x}2\mathrm{e}^x = 2\mathrm{e}^x$

3. $\frac{\mathrm{d}}{\mathrm{d}x}10\mathrm{e}^x = 10\mathrm{e}^x$

4. $\frac{\mathrm{d}}{\mathrm{d}x}(5x^2+\mathrm{e}^x) = 10x+\mathrm{e}^x$

5. $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x+3x^2+x^3) = \mathrm{e}^x+6x+3x^2$

6. $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{5x} = 5\mathrm{e}^{5x}$

7. $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{4x} = 4\mathrm{e}^{4x}$

8. $\frac{\mathrm{d}}{\mathrm{d}x}3\mathrm{e}^{4x} = 12\mathrm{e}^{4x}$

9. $\frac{\mathrm{d}}{\mathrm{d}x}3\mathrm{e}^{2x} = 6\mathrm{e}^{2x}$

10. $\frac{\mathrm{d}}{\mathrm{d}x}5\mathrm{e}^{4x} = 20\mathrm{e}^{4x}$

11. $\frac{\mathrm{d}}{\mathrm{d}x}(2\mathrm{e}^{3x}+3\mathrm{e}^{2x}) = 6\mathrm{e}^{3x}+6\mathrm{e}^{2x} = 6\mathrm{e}^{2x}(\mathrm{e}^x+1)$

12. $\frac{\mathrm{d}}{\mathrm{d}x}(4e^{3x}+x^4-2) = 12\mathrm{e}^{3x}+4x^3$

13. $\frac{\mathrm{d}}{\mathrm{d}x}e^{2x-4} = 2e^{2x-4}$

14. $\frac{\mathrm{d}}{\mathrm{d}x}e^{3x+1} = 3e^{3x+1}$

15. $\frac{\mathrm{d}}{\mathrm{d}x}e^{x^3} = 3x^2e^{x^3}$

16. $\frac{\mathrm{d}}{\mathrm{d}x}e^{x^2+5x-1} = (2x+5)e^{x^2+5x-1}$

17. Product rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}x^2\mathrm{e}^x &= 2x\mathrm{e}^x + x^2\mathrm{e}^x \\ &= x\mathrm{e}^x(2+x) \end{aligned}$$

18. Sum and product rules:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(x+x\mathrm{e}^x) &= 1+(\mathrm{e}^x+x\mathrm{e}^x) \\ &= 1+\mathrm{e}^x(1+x) \end{aligned}$$

19. Product rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}x^3\mathrm{e}^x &= 3x^2\mathrm{e}^x + x^3\mathrm{e}^x \\ &= x^2\mathrm{e}^x(3+x) \end{aligned}$$

20. $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^3+\mathrm{e}^x) = 3x^2+\mathrm{e}^x$

21. Product rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}x\mathrm{e}^{2x} &= \mathrm{e}^{2x} + 2x\mathrm{e}^{2x} \\ &= \mathrm{e}^{2x}(1+2x) \end{aligned}$$

22. Product rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}2x\mathrm{e}^x &= 2\mathrm{e}^x + 2x\mathrm{e}^x \\ &= 2\mathrm{e}^x(1+x) \end{aligned}$$

23. Quotient rule:
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{e}^x}{x} &= \frac{x\mathrm{e}^x - \mathrm{e}^x}{x^2}\\ &= \frac{\mathrm{e}^x(x-1)}{x^2}\end{aligned}$$

24. Product rule:
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x(1+x) &= \mathrm{e}^x(1+x) + \mathrm{e}^x\\ &= \mathrm{e}^x(2+x)\end{aligned}$$

25. Product rule:
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x(1+x)^5 &= \mathrm{e}^x(1+x)^5 + 5\mathrm{e}^x(1+x)^4\\ &= \mathrm{e}^x(1+x)^4(1+x+5)\\ &= \mathrm{e}^x(1+x)^4(6+x)\end{aligned}$$

26. Product rule:
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x(10-x)^4 &= \mathrm{e}^x(10-x)^4 + (-4)\mathrm{e}^x(10-x)^3\\ &= \mathrm{e}^x(10-x)^3(10-x-4)\\ &= \mathrm{e}^x(10-x)^3(6-x)\end{aligned}$$

27. $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\mathrm{e}^x} = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{-x} = -\mathrm{e}^{-x} = -\frac{1}{\mathrm{e}^x}$

28. $$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x + 2\mathrm{e}^{2x}$$
At $x = 1$,
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 6(1) + 2\mathrm{e}^{2(1)}\\ &= 6 + 2\mathrm{e}^2\end{aligned}$$

29. $$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x\mathrm{e}^{2x} + 2x^2\mathrm{e}^{2x}\\ &= 2x\mathrm{e}^{2x}(1+x)\end{aligned}$$
At $x = 1$,
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2(1)\mathrm{e}^{2(1)}(1+1)\\ &= 4\mathrm{e}^2\end{aligned}$$

30. $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{e^x}$ (see number 27 above)
$$\begin{aligned}-\frac{1}{\mathrm{e}^x} &= -\mathrm{e}\\ \frac{1}{\mathrm{e}^x} &= \mathrm{e}\\ 1 &= \mathrm{e}^{1+x}\\ 1+x &= \ln 1\\ &= 0\\ x &= -1\\ y &= \frac{1}{\mathrm{e}^{-1}}\\ &= \mathrm{e}\end{aligned}$$
The coordinates are $(-1, \mathrm{e})$.

31. First differentiate using the quotient rule:
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x\mathrm{e}^x - 2\mathrm{e}^x}{(2x)^2}\\ &= \frac{2\mathrm{e}^x(x-1)}{4x^2}\\ &= \frac{\mathrm{e}^x(x-1)}{2x^2}\end{aligned}$$
Now find where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$:
$$\begin{aligned}\frac{\mathrm{e}^x(x-1)}{2x^2} &= 0\\ \mathrm{e}^x(x-1) &= 0\\ x-1 &= 0\\ x &= 1\\ y &= \frac{\mathrm{e}^1}{2(1)}\\ &= \frac{\mathrm{e}}{2}\end{aligned}$$

32. $$\begin{aligned}\frac{\mathrm{d}R}{\mathrm{d}x} &= 10\,000 \times -0.5\mathrm{e}^{-0.5x}\\ &= -5\,000\mathrm{e}^{-0.5x}\ \$/\mathrm{wk}\end{aligned}$$

Exercise 7C

1. $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$$

2. $\log_e 2x = \log_e 2 + \log_e x$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 + \frac{1}{x} = \frac{1}{x}$$

3. $$\frac{\mathrm{d}y}{\mathrm{d}x} = 10x + \frac{1}{x}$$

4. $$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \mathrm{e}^x + \frac{1}{x}$$

5. Using the chain rule,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3x+2} \times 3 = \frac{3}{3x+2}$$

6. $\log_e x^2 = 2\log_e x$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x}$$

7. $\log_e(\sqrt[3]{x}) = \frac{1}{3}\log_e x$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3x}$$

8. $\log_e(3\sqrt{x}) = \log_e 3 + \log_e \sqrt{x} = \log_e 3 + \frac{1}{2}\log_e x$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2x}$$

9. $\log_e \frac{x}{5} = \log_e x - \log_e 5$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$$

10. Using the chain rule with $u = x(x+3)$,
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{x(x+3)}\left[(x+3)+x\right] \\ &= \frac{2x+3}{x(x+3)}\end{aligned}$$

11. Using the chain rule with $u = x^2 + x - 12$,
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{x^2+x-12}(2x+1) \\ &= \frac{2x+1}{x^2+x-12}\end{aligned}$$

12. Using the product rule:
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \log_e(x) + x\frac{1}{x} \\ &= \log_e(x) + 1\end{aligned}$$

13. Using the chain rule with $u = \log_e x$,
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3(\log_e x)^2\frac{1}{x} \\ &= \frac{3(\log_e x)^2}{x}\end{aligned}$$

14. Using the chain rule with $u = \frac{1}{x}$,
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\frac{1}{x}}\left(-\frac{1}{x^2}\right) \\ &= x\left(-\frac{1}{x^2}\right) \\ &= -\frac{1}{x}\end{aligned}$$
The above works just fine, but it's simpler to use a log law first:
$$\begin{aligned}y &= \log_e \frac{1}{x} \\ &= -\log_e x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{1}{x}\end{aligned}$$
It's not uncommon for there to be more than one way to do a problem, and it's similarly not uncommon to realise the simple approach only after you've done the more complicated one.

15. Using the chain rule with $u = \log_e x$,
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{1}{(\log_e x)^2} \times \frac{1}{x} \\ &= -\frac{1}{x(\log_e x)^2}\end{aligned}$$

16. Using the product rule,
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \mathrm{e}^x \log_e x + \mathrm{e}^x\frac{1}{x} \\ &= \mathrm{e}^x\left(\log_e x + \frac{1}{x}\right)\end{aligned}$$

17. Using the quotient rule,
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(\frac{1}{x})(x) - (\log_e x)(1)}{x^2} \\ &= \frac{1-\log_e x}{x^2}\end{aligned}$$

18. Using the chain rule with $u = 1 + \log_e x$,
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 3(1+\log_e x)^2(\frac{1}{x}) \\ &= \frac{3(1+\log_e x)^2}{x}\end{aligned}$$

19. Using the chain rule with $u = 3x^2 + 16x + 15$. Note that we can find $\frac{\mathrm{d}u}{\mathrm{d}x}$ using the product rule, but it's probably simpler to simply expand it: $u = x^3 + 8x^2 + 15x$.
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{x(x+5)(x+3)}(3x^2+16x+15) \\ &= \frac{3x^2+16x+15}{x(x+5)(x+3)}\end{aligned}$$

Again, however, this is simpler if we use log laws first:

$$\begin{aligned} y &= \log_e[x(x+5)(x+3)] \\ &= \log_e x + \log_e(x+5) + \log_e(x+3) \\ \frac{dy}{dx} &= \frac{1}{x} + \frac{1}{x+5} + \frac{1}{x+3} \end{aligned}$$

Students are encouraged to convince themselves that these two solutions are equivalent.

20. First simplify using log laws ...

$$\begin{aligned} y &= \log_e \frac{x+1}{x+3} \\ &= \log_e(x+1) - \log_e(x+3) \\ \frac{dy}{dx} &= \frac{1}{x+1} - \frac{1}{x+3} \end{aligned}$$

21. First simplify using log laws then use the chain rule with $u = x^2 + 5$:

$$\begin{aligned} y &= \log_e\left[(x^2+5)^4\right] \\ &= 4\log_e(x^2+5) \\ \frac{dy}{dx} &= 4\left(\frac{1}{x^2+5}\right)(2x) \\ &= \frac{8x}{x^2+5} \end{aligned}$$

22. Take as far as we can with log laws first, then differentiation is trivial.

$$\begin{aligned} y &= \log_e \frac{x}{x^2-1} \\ &= \log_e x - \log_e(x^2-1) \\ &= \log_e x - \log_e[(x-1)(x+1)] \\ &= \log_e x - \log_e(x-1) - \log_e(x+1) \\ \frac{dy}{dx} &= \frac{1}{x} - \frac{1}{x-1} - \frac{1}{x+1} \end{aligned}$$

23.
$$\begin{aligned} y &= \log_e \frac{(x+2)^3}{x-2} \\ &= \log_e[(x+2)^3] - \log_e(x-2) \\ &= 3\log_e(x+2) - \log_e(x-2) \\ \frac{dy}{dx} &= \frac{3}{x+2} - \frac{1}{x-2} \end{aligned}$$

24. $\frac{dy}{dx} = \frac{1}{x}$ so when $x = 1$, $\frac{dy}{dx} = \frac{1}{1} = 1$.

25. $\frac{dy}{dx} = \log_e(x) + 1$ (see question 12) so when $x = e^2$,

$$\begin{aligned} \frac{dy}{dx} &= \log_e(e^2) + 1 \\ &= 2 + 1 \\ &= 3 \end{aligned}$$

26. $y' = 6x + \frac{1}{x}$ so when $x = 1$, $y' = 6 \times 1 + \frac{1}{1} = 7$.

27. $y' = \frac{-2(1-\log_e x)}{x^2}$ (compare question 17) so when $x = 1$,

$$\begin{aligned} y' &= \frac{-2(1-\log_e(1))}{(1)^2} \\ &= \frac{-2(1-0)}{1} \\ &= -2 \end{aligned}$$

28. $y' = \frac{1}{x}$ so

$$\begin{aligned} \frac{1}{x} &= 0.25 \\ x &= 4 \\ y &= \ln 4 \end{aligned}$$

The coordinates of the one point with a gradient of 0.25 are $(4, \ln 4)$.

29. $y' = \frac{2}{x}$ (see question 6) so

$$\begin{aligned} \frac{2}{x} &= 4 \\ x &= 0.5 \\ y &= \ln 0.25 \end{aligned}$$

The coordinates of the one point with a gradient of 4 are $(0.5, \ln 0.25)$.

30. Using the chain rule we obtain $y' = \frac{6}{6x-5}$ so

$$\begin{aligned} \frac{6}{6x-5} &= 0.24 \\ \frac{1}{6x-5} &= 0.04 \\ 6x-5 &= \frac{1}{0.04} \\ &= 25 \\ x &= 5 \\ y &= \ln(6(5)-5) \\ &= \ln 25 \end{aligned}$$

The coordinates of the one point with a gradient of 0.24 are $(5, \ln 25)$.

31. Differentiating:

$$\begin{aligned} y &= \ln x + \ln(x+3) \\ y' &= \frac{1}{x} + \frac{1}{x+3} \\ \frac{1}{x} + \frac{1}{x+3} &= 0.5 \\ (x+3) + x &= 0.5x(x+3) \\ 2x+3 &= 0.5(x^2+3x) \\ 4x+6 &= x^2+3x \\ x^2+3x-4x-6 &= 0 \\ x^2-x-6 &= 0 \\ (x-3)(x+2) &= 0 \\ \text{either} \quad x &= 3 \\ y &= \ln[3(3+3)] \\ &= \ln 18 \\ \text{or} \quad x &= -2 \\ y &= \ln[-2(-2+3)] \\ &= \ln(-2) \end{aligned}$$

But $\ln(-2)$ is undefined so we have only one point with gradient 0.5 having coordinates $(3, \ln 18)$.

32. $y' = \frac{1}{x}$ so the gradient of the curve at $(1,0)$ is $m = \frac{1}{1} = 1$, then the tangent line is

$$\begin{aligned} (y - y_0) &= m(x - x_0) \\ y - 0 &= 1(x-1) \\ y &= x - 1 \end{aligned}$$

33. The gradient is $m = \frac{1}{\mathrm{e}}$ so the tangent line is

$$\begin{aligned} (y - y_0) &= m(x - x_0) \\ y - 1 &= \frac{1}{\mathrm{e}}(x - \mathrm{e}) \\ &= \frac{x}{\mathrm{e}} - 1 \\ y &= \frac{x}{\mathrm{e}} \end{aligned}$$

Miscellaneous Exercise 7

1. (a) It's a sine curve with amplitude 1 and period of 180° so

$$y = 1\sin 2x$$

   (b) It's a cosine curve with amplitude 2 and period of 360° so

$$y = 2\cos x$$

   (c) It's an inverted sine with amplitude 2 and period 360° so

$$y = -2\sin x$$

2. (a) $\mathrm{f}(x)$ is not differentiable at $x = a$ because it has a different gradient as we approach a from the left than when we approach from the right.

   $\mathrm{f}(x)$ is not differentiable at $x = c$ because it is not continuous at that point.

   $\mathrm{f}(x)$ is differentiable everywhere else (because linear and quadratic functions are always differentiable).

   (b) $\mathrm{f}'(x)$ has a constant positive value for $x < a$, is undefined at $x = a$, is linear with positive gradient for $a < x < c$, passing through 0 when $x = b$, is undefined for $x = c$ and is zero for $x > c$.

   See Sadler for a sketch of the graph.

3. Starting with the R.H.S.

$$\begin{aligned} \frac{2\tan\theta}{\tan^2\theta + 1} &= \frac{2\frac{\sin\theta}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta} + 1} \\ &= \frac{2\frac{\sin\theta}{\cos\theta}}{\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta}} \\ &= \frac{2\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}} \\ &= 2\frac{\sin\theta}{\cos\theta} \times \cos^2\theta \\ &= 2\sin\theta\cos\theta \\ &= \sin 2\theta \\ &= \text{L.H.S.} \end{aligned}$$

□

4. (a) Chain rule:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 4(3x^2+5)^3(6x) \\ &= 24x(3x^2+5)^3 \end{aligned}$$

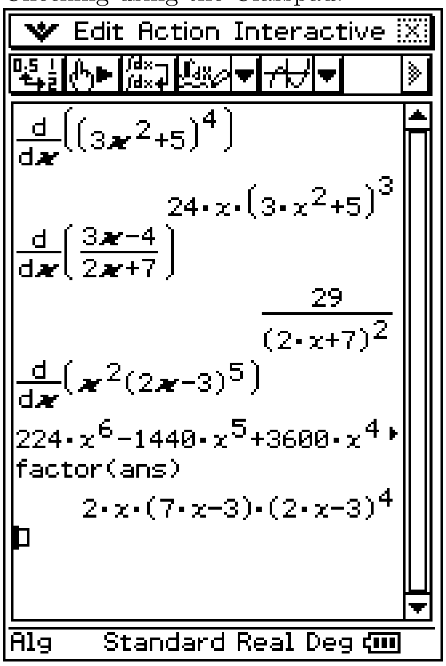
(b) Quotient rule:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3(2x+7)-2(3x-4)}{(2x+7)^2}\\ &= \frac{6x+21-6x+8}{(2x+7)^2}\\ &= \frac{29}{(2x+7)^2}\end{aligned}$$

(c) Product and chain rules:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x(2x-3)^5 + x^2[5(2x-3)^4](2)\\ &= 2x(2x-3)^5 + 10x^2(2x-3)^4\\ &= 2x(2x-3)^4(2x-3\ +5x)\\ &= 2x(2x-3)^4(7x-3)\end{aligned}$$

Checking using the Classpad:



5. (a) $$\begin{aligned}3x^2+11x-4 &= 3x^2+12x-x-4\\ &= 3x(x+4)-(x+4)\\ &= (x+4)(3x-1)\end{aligned}$$

(b) $$\begin{aligned}\lim_{x\to-4}\frac{3x^2+11x-4}{x+4} &= \lim_{x\to-4}\frac{(x+4)(3x-1)}{x+4}\\ &= \lim_{x\to-4}(3x-1)\\ &= 3(-4)-1\\ &= -13\end{aligned}$$

6. Continuity at $x=3$:

$$\begin{aligned}\lim_{x\to3^-}\mathrm{f}(x) &= \lim_{x\to3^+}\mathrm{f}(x)\\ \lim_{x\to3^-}(2x+5) &= \lim_{x\to3^+}(ax+2)\\ 11 &= 3a+2\\ a &= 3\end{aligned}$$

Continuity at $x=10$:

$$\begin{aligned}\lim_{x\to10^-}\mathrm{f}(x) &= \lim_{x\to10^+}\mathrm{f}(x)\\ \lim_{x\to10^-}(3x+2) &= \lim_{x\to10^+}(4x+b)\\ 32 &= 40+b\\ b &= -8\end{aligned}$$

7. a represents the $x-$value where the function is discontinuous, i.e. where the denominator goes to zero:

$$a=3$$

b represents the $y-$value that the function approaches when $x\to\infty$. Using the principle of dominant powers of x,

$$b=\lim_{x\to\infty}\frac{2x+12}{x-3}=\frac{2x}{x}=2$$

Point C is the $x-$intercept, i.e.

$$\begin{aligned}\frac{2x+12}{x-3} &= 0\\ 2x+12 &= 0\\ x &= -6\end{aligned}$$

so point C has coordinates $(-6,0)$.

Point D is the $y-$intercept, i.e.

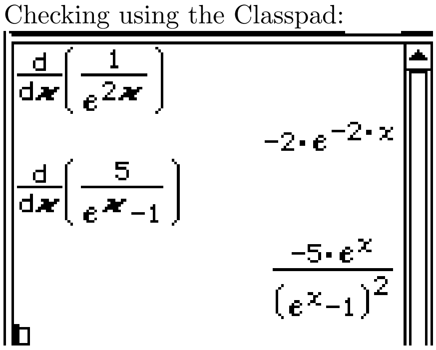
$$\begin{aligned}y &= \frac{2(0)+12}{(0)-3}\\ &= -4\end{aligned}$$

so point D has coordinates $(0,-4)$.

8. (a) $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\mathrm{e}^{2x}} &= \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{-2x}\\ &= -2\mathrm{e}^{-2x}\\ &= -\frac{2}{\mathrm{e}^{2x}}\end{aligned}$$

(b) $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\frac{5}{\mathrm{e}^x-1} &= \frac{0-5\mathrm{e}^x}{(\mathrm{e}^x-1)^2}\\ &= -\frac{5\mathrm{e}^x}{(\mathrm{e}^x-1)^2}\end{aligned}$$

Checking using the Classpad:



9. $$\begin{aligned}\lim_{x\to2^-}\mathrm{f}(x) &= 3(2)\\ &= 6\\ \lim_{x\to2^+}\mathrm{f}(x) &= (2)+2\\ &= 4\end{aligned}$$

The limit does not exist.

10.
$$\begin{aligned}\lim_{x\to1^-} \mathrm{g}(x) &= 1+2\\ &= 3\\ \lim_{x\to1^+} \mathrm{g}(x) &= 3(1)\\ &= 3\\ \therefore\quad \lim_{x\to1} \mathrm{g}(x) &= 3\end{aligned}$$

11. (a)
$$\begin{aligned}\lim_{x\to0^-} \mathrm{f}(x) &= \frac{2(0)+3}{0-1}\\ &= -3\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x\to0^+} \mathrm{f}(x) &= \frac{0-6}{0+2}\\ &= -3\end{aligned}$$

(c)
$$\lim_{x\to0} \mathrm{f}(x) = -3$$

(d)
$$\begin{aligned}\lim_{x\to-\infty} \mathrm{f}(x) &= \lim_{x\to-\infty} \frac{2x+3}{x-1}\\ &= 2\end{aligned}$$

(e)
$$\begin{aligned}\lim_{x\to\infty} \mathrm{f}(x) &= \lim_{x\to-\infty} \frac{x-6}{x+1}\\ &= 1\end{aligned}$$

12. (a) The absolute value function is continuous, so the limit is equal to the value of the function:
$$\lim_{x\to0}(|x|+3) = |0|+3 = 3$$

(b) The limit does not exist. The limit from the left is -1 while from the right it is 1.

(c) The function is continuous at $x = 0$ (its only discontinuity is at $x = 3$) so
$$\lim_{x\to0} = \frac{|0-3|}{0-3} = -1$$

(d) The limit does not exist. The limit from the left is -1 while from the right it is 1.

13. We need only concern ourselves with continuity and differentiabilty at $x = 2$ as everywhere else we are dealing with polynomial functions.

For continuity,
$$\begin{aligned}-23(-2)-28 &= a(-2)+b((-2)^2)\\ -2a+4b &= 18\\ -a+2b &= 9\end{aligned}$$

For differentiability,
$$\begin{aligned}-23 &= a+2b(-2)\\ a-4b &= -23\end{aligned}$$

Solving simultaneously
$$\begin{aligned}-2b &= -14\\ b &= 7\\ -a+2b &= 9\\ -a+14 &= 9\\ a &= 5\end{aligned}$$

14.
$$\begin{aligned}z-2+7\mathrm{i} &= \pm5\mathrm{i}\\ x &= 2-12\mathrm{i}\\ \text{or}\quad x &= 2-2\mathrm{i}\end{aligned}$$

15.
$$\begin{aligned}\sin 105^\circ &= \sin(60^\circ+45^\circ)\\ &= \sin 60^\circ\cos 45^\circ + \cos 60^\circ \sin 45^\circ\\ &= \frac{\sqrt3}{2}\frac{\sqrt2}{2} + \frac12\frac{\sqrt2}{2}\\ &= \frac{\sqrt2(\sqrt3+1)}{4}\end{aligned}$$

16. Refer to the sketches in Sadler's solutions.

(a) The gradient is positive everywhere and tends towards zero as $x \to \pm\infty$. As we approach $x = a$ from the left or the right the gradient increases without bound.

(b) Here the gradient is negative everywhere and tends towards zero as $x \to \pm\infty$. As we approach $x = a$ from the left or the right the gradient decreases without bound.

(c) The gradient is positive for $x < a$, as $x \to -\infty$ so $\mathrm{h}'(x) \to 0$ and as $x \to a^-$ so $\mathrm{h}'(x) \to \infty$.
The gradient is negative for $x > a$, as $x \to \infty$ so $\mathrm{h}'(x) \to 0$ and as $x \to a^+$ so $\mathrm{h}'(x) \to -\infty$.

17. The position of the boat after 3 hours (i.e. at noon) is $3(12\mathbf{i}+4\mathbf{j}) = (36\mathbf{i}+12\mathbf{j})$km. The displacement from there to the harbour at A is
$$(42\mathbf{i}+9\mathbf{j})-(36\mathbf{i}+12\mathbf{j}) = (6\mathbf{i}-3\mathbf{j})\text{km}$$

Let the boat's velocity be $\mathbf{v} = (a\mathbf{i}+b\mathbf{j})$km/h. To steam directly to A, the vector sum of this velocity and that of the wind and current must be a scalar multiple of this displacement:
$$(a\mathbf{i}+b\mathbf{j})+(6\mathbf{i}+2\mathbf{j}) = k(6\mathbf{i}-3\mathbf{j})$$

(where $k = \frac{1}{t}$ for t the time in hours)
$$\begin{aligned}(a\mathbf{i}+b\mathbf{j}) &= k(6\mathbf{i}-3\mathbf{j})-(6\mathbf{i}+2\mathbf{j})\\ a &= 6k-6\\ b &= -3k-2\end{aligned}$$

From the boats speed of 10km/h,
$$\begin{aligned}a^2+b^2 &= 10^2\\ (6k-6)^2+(-3k-2)^2 &= 100\\ 36k^2-72k+36+9k^2+12k+4-100 &= 0\\ 45k^2-60k-60 &= 0\\ 3k^2-4k-4 &= 0\\ 3k^2-6k+2k-4 &= 0\\ 3k(k-2)+2(k-2) &= 0\\ (k-2)(3k+2) &= 0\\ k &= 2\end{aligned}$$

(we can ignore the other root because it gives negative time)

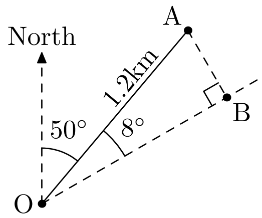
$$\begin{aligned} a &= 6k - 6 \\ &= 6 \\ b &= -3k - 2 \\ &= -8 \\ t &= \frac{1}{2} \end{aligned}$$

The velocity of the boat should be set to $(6\mathbf{i} - 8\mathbf{j})$km/h. The boat will arrive after half an hour, i.e. at 12:30pm.

Chapter 8

Exercise 8A

1. Let O be the starting point, A be the first checkpoint and B be the point along the competitor's path that is nearest A.

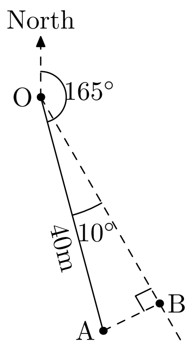


To be the nearest point, B must be such that AB is perpendicular to OB. This means that OAB is a right angled triangle.

$$\begin{aligned}\sin 8^\circ &= \frac{AB}{OA}\\ &= \frac{AB}{1.2}\\ AB &= 1.2\sin 8^\circ\\ &= 0.167\text{km}\end{aligned}$$

The competitor comes within about 170m of the checkpoint.

2. Let O be the position of the batsman, A be the position of the fielder and B be the point along the ball's path that is nearest A.

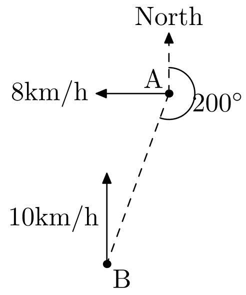


To be the nearest point, B must be such that AB is perpendicular to OB. This means that OAB is a right angled triangle.

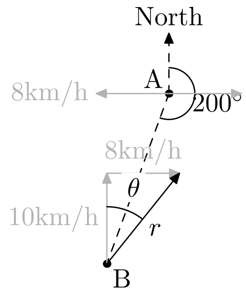
$$\begin{aligned}\sin 10^\circ &= \frac{AB}{OA}\\ &= \frac{AB}{40}\\ AB &= 40\sin 10^\circ\\ &= 6.945\text{m}\end{aligned}$$

The ball passes about 6.9m from the fielder who can not be confident of stopping it.

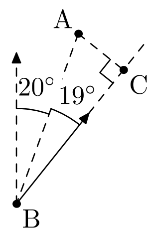
3. Impose a velocity of 8km/h due east to consider the situation from the point of view of an observer on vessel A.



becomes



$$\begin{aligned}\tan\theta &= \frac{8}{10}\\ \theta &= \tan^{-1} 0.8\\ &= 39^\circ\end{aligned}$$

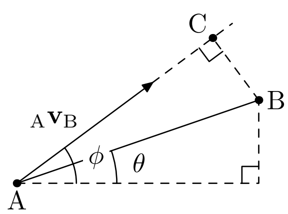


$$\begin{aligned}\sin 19^\circ &= \frac{AC}{AB}\\ AC &= AB\sin 19^\circ\\ &= 2\sin 19^\circ\\ &= 0.64\text{km}\end{aligned}$$

The vessels pass within about 640m of each other.

4. First with vectors:

$$\begin{aligned}\overrightarrow{\mathrm{AB}} &= \mathbf{r}_{\mathrm{B}} - \mathbf{r}_{\mathrm{A}}\\ &= (22\mathbf{i} + 21\mathbf{j}) - (-10\mathbf{i} + 10\mathbf{j})\\ &= (32\mathbf{i} + 11\mathbf{j})\text{km}\\ {}_{\mathrm{A}}\mathbf{v}_{\mathrm{B}} &= \mathbf{v}_{\mathrm{A}} - \mathbf{v}_{\mathrm{B}}\\ &= (15\mathbf{i} + 10\mathbf{j}) - (-5\mathbf{i})\\ &= (20\mathbf{i} + 10\mathbf{j})\text{km/h}\end{aligned}$$



$$\begin{aligned}
\tan\theta &= \frac{11}{32}\\
\theta &= \tan^{-1}\frac{11}{32}\\
\tan\phi &= \frac{10}{20}\\
\phi &= \tan^{-1}\frac{1}{2}\\
\sin(\phi-\theta) &= \frac{\text{BC}}{\text{BA}}\\
\text{BC} &= \text{BA}\sin(\phi-\theta)\\
&= \sqrt{32^2+11^2}\sin(\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{11}{32})\\
&= 4.47\text{km}\\
\text{AC} &= \sqrt{32^2+11^2}\cos(\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{11}{32})\\
&= 33.54\text{km}\\
t &= \frac{33.54}{|_{\text{A}}\mathbf{v}_{\text{B}}}\\
&= \frac{33.54}{\sqrt{20^2+10^2}}\\
&= 1.5\text{hours}
\end{aligned}$$

The ships pass within about 4.5km of each other after 1.5 hours, i.e. at 11:30am.

Now using a different approach:

$$\begin{aligned}
\mathbf{r}_{\text{A}}(t) &= (-10\mathbf{i}+10\mathbf{j})+t(15\mathbf{i}+10\mathbf{j})\\
&= (-10+15t)\mathbf{i}+(10+10t)\mathbf{j}\\
\mathbf{r}_{\text{B}}(t) &= (22\mathbf{i}+21\mathbf{j})+t(-5\mathbf{i})\\
&= (22-5t)\mathbf{i}+21\mathbf{j}
\end{aligned}$$

let d be the distance AB

$$\begin{aligned}
d(t) &= |\mathbf{r}_{\text{B}}(t)-\mathbf{r}_{\text{A}}(t)|\\
&= |[(22-5t)\mathbf{i}+21\mathbf{j}]\\
&\qquad -[(-10+15t)\mathbf{i}+(10+10t)\mathbf{j}]\\
&= |(32-20t)\mathbf{i}+(11-10t)\mathbf{j}|\\
d^2(t) &= (32-20t)^2+(11-10t)^2\\
&= 1024-1280t+400t^2\\
&\qquad +121-220t+100t^2\\
&= 500t^2-1500t+1145
\end{aligned}$$

This has a minimum when $t=\frac{-b}{2a}$, i.e.

$$\begin{aligned}
t &= \frac{1500}{2\times 500}\\
&= 1.5\\
d^2(1.5) &= 500(1.5)^2-1500(1.5)+1145\\
&= 1125-2250+1145\\
&= 20\\
d &= \sqrt{20}\\
&= 2\sqrt{5}\\
&\approx 4.47\text{km}
\end{aligned}$$

For finding the minimum distance there is not much between the two methods, but if you also need to find the time, the quadratic approach may be a little simpler.

5. Using the quadratic approach:

$$\begin{aligned}
\mathbf{r}_{\text{A}}(t) &= 10t\mathbf{i}+(20+10t)\mathbf{j}\\
\mathbf{r}_{\text{B}}(t) &= (24+t)\mathbf{i}+(18+7t)\mathbf{j}\\
\overrightarrow{\text{AB}}(t) &= (24+t)\mathbf{i}+(18+7t)\mathbf{j}\\
&\qquad -[10t\mathbf{i}+(20+10t)\mathbf{j}]\\
&= (24-9t)\mathbf{i}+(-2-3t)\mathbf{j}\\
d(t) &= |\overrightarrow{\text{AB}}(t)|\\
&= |(24-9t)\mathbf{i}+(-2-3t)\mathbf{j}|\\
d^2(t) &= (24-9t)^2+(-2-3t)^2\\
&= 576-432t+81t^2\\
&\qquad +4+12t+9t^2\\
&= 90t^2-420t+580
\end{aligned}$$

minimum is where

$$\begin{aligned}
t &= \frac{420}{2\times 90}\\
&= \frac{7}{3}\\
d\left(\frac{7}{3}\right) &= \sqrt{90\left(\frac{7}{3}\right)^2-420\left(\frac{7}{3}\right)+580}\\
&= 3\sqrt{10}
\end{aligned}$$

The minimum distance will be $3\sqrt{10}\approx 9.5$km. (This will occur after 2 hours and 20 minutes, i.e. at 2:20pm.)

6. The position vectors of A and B at time t are:

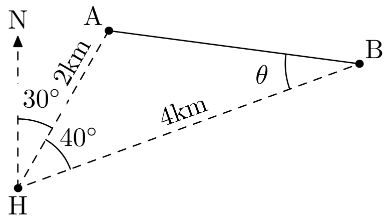
$$\begin{aligned}
\mathbf{r}_{\text{A}} &= (11-5t)\mathbf{i}+(5+3t)\mathbf{j}\\
\mathbf{r}_{\text{B}} &= (-7.5+3t)\mathbf{i}+(9.5-3t)\mathbf{j}\\
\overrightarrow{\text{AB}}(t) &= (-18.5+8t)\mathbf{i}+(4.5-6t)\mathbf{j}\\
d^2(t) &= (-18.5+8t)^2+(4.5-6t)^2\\
&= 100t^2-350t+362.5
\end{aligned}$$

minimum where

$$\begin{aligned}
t &= \frac{350}{2\times 200}\\
&= 1.75\text{s}\\
d(1.75) &= \sqrt{100(1.75)^2-350(1.75)+362.5}\\
&= 7.5\text{m}
\end{aligned}$$

The least distance is 7.5m and occurs at $t=1.75$ seconds.

7. First find $\overrightarrow{\text{AB}}$:



Find the length AB using the cosine rule:

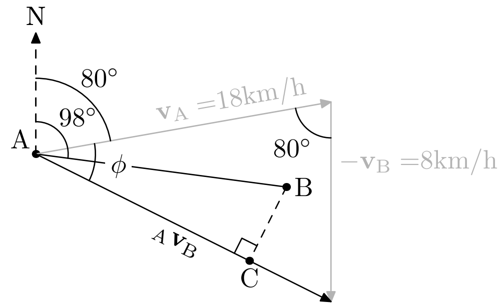
$$\begin{aligned} AB &= \sqrt{4^2 + 2^2 - 2 \times 4 \times 2 \cos 40^\circ} \\ &= 2.783\text{km} \end{aligned}$$

Find an angle using the sine rule. The angle θ is unambiguous (it can't be obtuse because it's not opposite the longest side of the triangle).

$$\begin{aligned} \frac{\sin\theta}{2} &= \frac{\sin 40^\circ}{2.783} \\ \theta &= \sin^{-1} \frac{2\sin 40^\circ}{2.783} \\ &= 27.51^\circ \end{aligned}$$

so the bearing from A to B is $70^\circ + 27.51 = 097.51^\circ$ (or 098° to the nearest degree).

The velocity of B relative to A:



Using the cosine rule

$$\begin{aligned} |{}_A\mathbf{v}_B| &= \sqrt{8^2 + 18^2 - 2 \times 8 \times 18 \cos 80^\circ} \\ &= 18.38 \end{aligned}$$

Using the sine rule

$$\begin{aligned} \frac{\sin\phi}{8} &= \frac{\sin 80^\circ}{18.38} \\ \phi &= \sin^{-1} \frac{8\sin 80^\circ}{18.38} \\ &= 25.37^\circ \\ \angle BAC &= \phi - 18 \\ &= 7.37^\circ \\ \sin \angle 7.37^\circ &= \frac{BC}{AB} \\ BC &= AB \sin \angle 7.37^\circ \\ &= 2.783 \sin \angle 7.37^\circ \\ &= 351\text{km} \\ &\approx 400\text{m} \end{aligned}$$

Exercise 8B

1. $\mathbf{a} \cdot \mathbf{b} = 5 \times 3 \cos 30^\circ = \frac{15\sqrt{3}}{2}$
2. $\mathbf{a}$ and $\mathbf{c}$ are perpendicular so $\mathbf{a} \cdot \mathbf{c} = 0$
3. $\mathbf{a} \cdot \mathbf{d} = 3 \times 2 \times \cos 180^\circ = -6$
4. $\mathbf{b} \cdot \mathbf{c} = 5 \times 4 \cos 60^\circ = 10$
5. $\mathbf{b} \cdot \mathbf{d} = 5 \times 2 \cos 150^\circ = -5\sqrt{3}$
6. $\mathbf{c}$ and $\mathbf{d}$ are perpendicular so $\mathbf{c} \cdot \mathbf{d} = 0$
7. $\mathbf{e}$ and $\mathbf{f}$ are perpendicular so $\mathbf{e} \cdot \mathbf{f} = 0$
8. $\mathbf{f} \cdot \mathbf{e} = \mathbf{e} \cdot \mathbf{f} = 0$
9. $\mathbf{e} \cdot \mathbf{e} = e^2 = 2^2 = 4$
10. $\mathbf{f} \cdot \mathbf{f} = f^2 = 3^2 = 9$
11. $\mathbf{f} \cdot \mathbf{g} = 4 \times 3 \cos 45^\circ = 6\sqrt{2}$
12. $\mathbf{g} \cdot \mathbf{f} = \mathbf{f} \cdot \mathbf{g} = 6\sqrt{2}$
13. $\mathbf{a} \cdot \mathbf{b} = 4 \times 6 \cos 60^\circ = 12$
14. $\mathbf{a} \cdot \mathbf{b} = 8 \times 6 \cos 120^\circ = -24$
15. If we draw the vectors tail to tail the angle is 60° so $\mathbf{a} \cdot \mathbf{b} = 2 \times 3 \cos 60^\circ = 3$
16. The vectors are perpendicular so the scalar product is 0.
17. $\mathbf{a} \cdot \mathbf{b} = 7 \times 10 \cos 150^\circ = -35\sqrt{3}$
18. $\mathbf{a} \cdot \mathbf{b} = 10 \times 20 \cos 135^\circ = -100\sqrt{2}$
19. (a) The magnitude of a vector is a scalar.
    (b) The dot product of two vectors is a scalar.
    (c) The sum of two vectors is a vector.
    (d) The difference of two vectors is a vector.

(e) $2\mathbf{b}$ is a vector so this is the sum of two vectors: a vector.

(f) This is the dot product of two vectors: it's a scalar.

(g) $(\mathbf{a}+\mathbf{b})$ is a vector, as is $(\mathbf{c}+\mathbf{d})$, so this is the dot product of two vectors: it's a scalar.

(h) Magnitude is always a scalar.

(i) $\lambda\mathbf{b}$ is a vector so this is the sum of two vectors: a vector.

(j) This is the dot product of two vectors: it's a scalar.

20. (a) $\mathbf{i}\cdot\mathbf{i} = |\mathbf{i}|^2 = 1^2 = 1$

(b) $\mathbf{i}$ and $\mathbf{j}$ are perpendicular so $\mathbf{i}\cdot\mathbf{j} = 0$

(c) $\mathbf{j}\cdot\mathbf{j} = |\mathbf{j}|^2 = 1^2 = 1$

21. (a)
$$\begin{aligned}&(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b}\\ &= a^2 - b^2\end{aligned}$$

(b)
$$\begin{aligned}&(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b}\\ &= a^2 + 2\mathbf{a}\cdot\mathbf{b} + b^2\end{aligned}$$

(c)
$$\begin{aligned}&(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a} - 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b}\\ &= a^2 - 2\mathbf{a}\cdot\mathbf{b} + b^2\end{aligned}$$

(d)
$$\begin{aligned}&(2\mathbf{a}+\mathbf{b})\cdot(2\mathbf{a}-\mathbf{b})\\ &= 2\mathbf{a}\cdot 2\mathbf{a} - 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot 2\mathbf{a} - \mathbf{b}\cdot\mathbf{b}\\ &= 4\mathbf{a}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b}\\ &= 4a^2 - b^2\end{aligned}$$

(e)
$$\begin{aligned}&(\mathbf{a}+3\mathbf{b})\cdot(\mathbf{a}-2\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot 2\mathbf{b} + 3\mathbf{b}\cdot\mathbf{a} - 3\mathbf{b}\cdot 2\mathbf{b}\\ &- \mathbf{a}\cdot\mathbf{a} - 2\mathbf{a}\cdot\mathbf{b} + 3\mathbf{b}\cdot\mathbf{a} - 6\mathbf{b}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} - 6\mathbf{b}\cdot\mathbf{b}\\ &= a^2 + \mathbf{a}\cdot\mathbf{b} - 6b^2\end{aligned}$$

(f)
$$\begin{aligned}&\mathbf{a}\cdot(\mathbf{a}-\mathbf{b}) + \mathbf{a}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a}\\ &= a^2\end{aligned}$$

22. L.H.S.:

$$\begin{aligned}&(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-2\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{a} - 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} - 2\mathbf{b}\cdot\mathbf{b}\\ &= a^2 - 0 + 0 - 2b^2\\ &= a^2 - 2b^2\\ &= \text{R.H.S.}\end{aligned}$$

□

23. (a) This is not true. Non-zero vectors can't be both equal and perpendicular.

(b) This is true. If vectors are perpendicular they have a zero dot product.

(c) This is not (necessarily) true. $ab = 0$ implies one (at least) of the vectors is the zero vector.

(d) This is true. $\mathbf{a}\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b} = a^2+0$.

24. (a) True. The vectors are perpendicular so their dot product is zero.

(b) True. This follows from (a).

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}-\mathbf{c}) &= 0\\ \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{c} &= 0\\ \mathbf{a}\cdot\mathbf{b} &= \mathbf{a}\cdot\mathbf{c}\end{aligned}$$

(c) Not true. In fact if this were true $(\mathbf{b}-\mathbf{c})$ would be the zero vector and it doesn't make much sense to talk about perpendicularity with the zero vector. There's an important point here: you can't simplify an expression by 'dividing' by a vector. Division by a vector is undefined. Unlike a scalar multiplication, you can not 'undo' a dot product by any inverse operation.

(d) True. If $\mathbf{a}$ is perpendicular with $(\mathbf{b}-\mathbf{c})$ it must also be perpendicular with its opposite.

25.
$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (x_1\mathbf{i}+y_1\mathbf{j})\cdot(x_2\mathbf{i}+y_2\mathbf{j})\\ &= x_1x_2\mathbf{i}\cdot\mathbf{i} + x_1y_2\mathbf{i}\cdot\mathbf{j} + x_2y_1\mathbf{i}\cdot\mathbf{j} + y_1y_2\mathbf{j}\cdot\mathbf{j}\\ &= x_1x_2(1) + x_1y_2(0) + x_2y_1(0) + y_1y_2(1)\\ &= x_1x_2 + y_1y_2\end{aligned}$$

26. (a)
$$\begin{aligned}\mathbf{a}\cdot(\mathbf{a}-\mathbf{b}) &= \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b}\\ &= a^2 - 0\\ &\neq 0\end{aligned}$$

Not true.

(b) Refer to question 21(a) for the first step.

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) &= a^2 - b^2\\ &\neq 0 \text{ unless } a = b\end{aligned}$$

Not necessarily true.

(c) Refer to question 21(b) for the first step.

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) &= a^2 + 2\mathbf{a}\cdot\mathbf{b} + b^2\\ &= a^2 + 0 + b^2\end{aligned}$$

True.

27. (a)
$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}||\mathbf{b}|\cos\theta\\ 7 &= 5\times 3\cos\theta\\ \cos\theta &= \frac{7}{15}\\ \theta &= \cos^{-1}\frac{7}{15}\\ &= 62°\end{aligned}$$

(b) $\mathbf{a}\cdot\mathbf{a} = a^2 = 5^2 = 25$

(c) $\mathbf{b}\cdot\mathbf{b} = b^2 = 3^2 = 9$

(d) Refer question 21(c) for the first step.

$$\begin{aligned}(\mathbf{a}-\mathbf{b})&\cdot(\mathbf{a}-\mathbf{b})\\ &= a^2 - 2\mathbf{a}\cdot\mathbf{b} + b^2\\ &= 5^2 - 2(7) + 3^2\\ &= 20\end{aligned}$$

(e) You may be tempted to use trigonometry and the angle between vectors, but there's a simpler approach. The dot product of any vector with itself is the square of its magnitude, so the magnitude must be the square root of the dot product:

$$\begin{aligned}|\mathbf{a}-\mathbf{b}| &= \sqrt{(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})}\\ &= \sqrt{20}\\ &= 2\sqrt{5}\end{aligned}$$

28.
$$\begin{aligned}\mathbf{p}\cdot\mathbf{q} &= (3\mathbf{a}+2\mathbf{b})\cdot(4\mathbf{a}-\mathbf{b})\\ &= 12a^2 - 3\mathbf{a}\cdot\mathbf{b} + 8\mathbf{a}\cdot\mathbf{b} - 2b^2\\ &= 12(3^2) - 3(0) + 8(0) - 2(2^2)\\ &= 108 - 8\\ &= 100\end{aligned}$$

29. The scalar product is, by definition, the product of two vectors. $(\mathbf{b}+\mathbf{c})$ and $(\mathbf{b}-\mathbf{c})$ are both vectors so $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$ and $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$ but $(\mathbf{b}\cdot\mathbf{c})$ is a scalar, so $\mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c})$ is an attempt to take the scalar product of a vector and a scalar, and so is meaningless.

30. (a)
$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$
The magnitudes are both positive, so if we take the absolute value we get
$$|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}||\mathbf{b}||\cos\theta|$$
and since $-1 \le \cos\theta \le 1$ it follows that
$$\begin{aligned}&0 \le |\cos\theta| \le 1\\ \therefore\quad &0 \le |\mathbf{a}||\mathbf{b}||\cos\theta| \le |\mathbf{a}||\mathbf{b}|\\ \therefore\quad &|\mathbf{a}\cdot\mathbf{b}| \le |\mathbf{a}||\mathbf{b}|\end{aligned}$$

□

(b) Refer to question 21(b) for the first step.

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) &= a^2 + 2\mathbf{a}\cdot\mathbf{b} + b^2\\ |\mathbf{a}+\mathbf{b}|^2 &= a^2 + 2\mathbf{a}\cdot\mathbf{b} + b^2\\ (|\mathbf{a}|+|\mathbf{b}|)^2 &= a^2 + 2|\mathbf{a}||\mathbf{b}| + b^2\end{aligned}$$

But

$$\begin{aligned}&\mathbf{a}\cdot\mathbf{b} \le |\mathbf{a}||\mathbf{b}|\\ \therefore\quad &2\mathbf{a}\cdot\mathbf{b} \le 2|\mathbf{a}||\mathbf{b}|\\ \therefore\quad &a^2 + 2\mathbf{a}\cdot\mathbf{b} + b^2 \le 2a^2 + |\mathbf{a}||\mathbf{b}| + b^2\\ \therefore\quad &|\mathbf{a}+\mathbf{b}|^2 \le (|\mathbf{a}|+|\mathbf{b}|)^2\end{aligned}$$

and since all the magnitudes are positive

$$\therefore\quad |\mathbf{a}+\mathbf{b}| \le |\mathbf{a}|+|\mathbf{b}|$$

□

Exercise 8C

1. (a) $\mathbf{a}\cdot\mathbf{b} = 3\times5 + (-2)\times6 = 15 - 12 = 3$
   (b) $\mathbf{b}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{b} = 3$
   (c) $\mathbf{a}\cdot\mathbf{c} = 3\times2 + (-2)\times(-1) = 6 + 2 = 8$
   (d) $\mathbf{b}\cdot\mathbf{c} = 5\times2 + 6\times(-1) = 10 - 6 = 4$

2. (a) $\mathbf{x}\cdot\mathbf{y} = 2\times5 + 3\times(-1) = 10 - 3 = 7$
   (b) $\mathbf{x}\cdot\mathbf{z} = 2\times4 + 3\times2 = 8 + 6 = 14$
   (c) $\mathbf{z}\cdot\mathbf{x} = \mathbf{x}\cdot\mathbf{z} = 14$
   (d) $\mathbf{y}\cdot\mathbf{z} = 5\times4 + (-1)\times2 = 20 - 2 = 18$

3. (a) $\mathbf{q}\cdot\mathbf{r} = 2\times5 + (-1)\times2 = 10 - 2 = 8$
   (b) $2\mathbf{q}\cdot3\mathbf{r} = 2\times2\times3\times5 + 2\times-1\times3\times2$
   $= 60 - 12 = 48$
   Alternatively $2\mathbf{q}\cdot3\mathbf{r} = 6\mathbf{q}\cdot\mathbf{r} = 6\times8 = 48$
   (c) $\mathbf{p}\cdot(\mathbf{q}+\mathbf{r}) = \langle3,1\rangle\cdot\langle2+5,-1+2\rangle$
   $= 3\times7 + 1\times1 = 22$
   (d) $\mathbf{p}\cdot(\mathbf{q}-\mathbf{r}) = \langle3,1\rangle\cdot\langle2-5,-1-2\rangle$
   $= 3\times-3 + 1\times-3 = -12$

4. (a) $2\times4 + 3\times-2 \ne 0$
   Vectors are not perpendicular.
   (b) $-2\times4 + 1\times-2 \ne 0$
   Vectors are not perpendicular.
   (c) $3\times2 + -1\times6 = 0$
   Vectors are perpendicular.
   (d) $12\times1 + -3\times4 = 0$
   Vectors are perpendicular.
   (e) $5\times-3 + 2\times7 \ne 0$
   Vectors are not perpendicular.

(f) $14 \times -4 + 8 \times 7 = 0$
Vectors are perpendicular.

5. (a) $3 \times 2 + 1 \times 4 = 10$
(b) $2 \times -2 + 4 \times -3 = -16$
(c) $3(2-2) + 1(4-3) = 1$
(d) $(3+2)(-2) + (1+4)(-3) = -25$

6. (a) $2(3+4) + (-1)(2-3) = 15$
(b) $(2+3)(4) + (-1+2)(-3) = 17$
(c) $3(2+4) + 2(-1-3) = 10$
(d) $$\begin{aligned}(2-3)(3-4)+&(-1-2)(2--3)\\ &= 1-15\\ &= -14\end{aligned}$$

7. (a) $\mathbf{a} \cdot \mathbf{b} = 2 \times 1 + 3 \times -4 = -10$
(b) $\mathbf{a} \cdot \mathbf{c} = 2 \times -4 + 3 \times 5 = 7$
(c) $\mathbf{b} + \mathbf{c} = (1-4)\mathbf{i} + (-4+5)\mathbf{j} = -3\mathbf{i} + \mathbf{j}$
(d) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = 2 \times -3 + 3 \times 1 = -3$
(e) $\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = -10 + 7 = -3$

8. (a) $|\mathbf{p}| = \sqrt{3^2 + 4^2} = 5$
(b) $|\mathbf{q}| = \sqrt{5^2 + (-12)^2} = 13$
(c) $\mathbf{p} \cdot \mathbf{q} = 3 \times 5 + 4 \times -12 = -33$
$$\begin{aligned}\mathbf{p} \cdot \mathbf{q} &= |\mathbf{p}||\mathbf{q}| \cos\theta\\ -33 &= 5 \times 13 \cos\theta\\ \cos\theta &= \frac{-33}{65}\\ \theta &= \cos^{-1} \frac{-33}{65}\\ &= 121°\end{aligned}$$

9. (a) $|\mathbf{c}| = \sqrt{7^2 + 7^2} = 7\sqrt{2}$
(b) $|\mathbf{d}| = \sqrt{15^2 + (-8)^2} = 17$
(c) $\mathbf{c} \cdot \mathbf{d} = 7 \times 15 + 7 \times -8 = 49$
$$\begin{aligned}\mathbf{c} \cdot \mathbf{d} &= |\mathbf{c}||\mathbf{d}| \cos\theta\\ 49 &= 7\sqrt{2} \times 17 \cos\theta\\ \cos\theta &= \frac{49}{119\sqrt{2}}\\ \theta &= \cos^{-1} \frac{7}{17 sqrt2}\\ &= 73°\end{aligned}$$

10. $$\begin{aligned}\mathbf{b} + \mathbf{c} &= (2+4)\mathbf{i} + (5-1)\mathbf{j}\\ &= 6\mathbf{i} + 4\mathbf{j}\\ \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= 2 \times 6 + (-3) \times 4\\ &= 0\end{aligned}$$
$\therefore$ $\mathbf{a}$ is perpendicular to $(\mathbf{b} + \mathbf{c})$. □

11. $$\begin{aligned}\mathbf{a} + 2\mathbf{b} &= (-2 + 2 \times 5)\mathbf{i} + (2 + 2 \times 2)\mathbf{j}\\ &= 8\mathbf{i} + 6\mathbf{j}\\ \mathbf{b} - 2\mathbf{c} &= (5 - 2 \times 4)\mathbf{i} + (2 - 2 \times -1)\mathbf{j}\\ &= -3\mathbf{i} + 4\mathbf{j}\\ (\mathbf{a} + 2\mathbf{b}) \cdot (\mathbf{b} - 2\mathbf{c}) &= 8 \times -3 + 6 \times 4\\ &= 0\end{aligned}$$
$\therefore$ $(\mathbf{a} + 2\mathbf{b})$ is perpendicular to $(\mathbf{b} - 2\mathbf{c})$. □

The approach used for finding the angle in questions 12 to 17 is identical.
$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}||\mathbf{b}| \cos\theta\\ \cos\theta &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\\ \theta &= \cos^{-1} \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\end{aligned}$$
There seems little point in working through each of these in detail here. Question 12 is done as a sample:

12. $$\begin{aligned}|\mathbf{a}| &= \sqrt{3^2 + 4^2}\\ &= 5\\ |\mathbf{b}| &= \sqrt{4^2 + 3^2}\\ &= 5\\ \mathbf{a} \cdot \mathbf{b} &= 3 \times 4 + 4 \times 3\\ &= 24\\ \theta &= \cos^{-1} \frac{24}{5 \times 5}\\ &= 16°\end{aligned}$$

18. For $\mathbf{a}$ and $\mathbf{b}$ to be parallel
$$\begin{aligned}\frac{3}{2} &= \frac{1}{2}\lambda\\ \lambda &= 8\end{aligned}$$
For $\mathbf{a}$ and $\mathbf{c}$ to be perpendicular
$$\begin{aligned}2\mu + 3 \times -5 &= 0\\ 2\mu &= 15\\ \mu &= 7.5\end{aligned}$$

19. For $\mathbf{d}$ and $\mathbf{e}$ to be perpendicular
$$\begin{aligned}w \times -1 + 1 \times 7 &= 0\\ w &= 7\end{aligned}$$
For $|\mathbf{d}| = |\mathbf{f}|$ with $x < 0$:
$$\begin{aligned}\sqrt{7^2 + 1^2} &= \sqrt{x^2 + 5^2}\\ 50 &= x^2 + 25\\ x^2 &= 25\\ x &= -5\end{aligned}$$

20. From our understanding of the vector equation of a line, we can say that L_1 and L_2 will be perpendicular if $\begin{pmatrix}-1\\3\end{pmatrix}$ and $\begin{pmatrix}6\\2\end{pmatrix}$ are perpendicular (since these vectors represent the direction of each line respectively).
$$\begin{aligned}\begin{pmatrix}-1\\3\end{pmatrix} \cdot \begin{pmatrix}6\\2\end{pmatrix} &= -1 \times 6 + 3 \times 2\\ &= 0\end{aligned}$$
$\therefore$ the lines are perpendicular. □

21. $$\begin{aligned}(a\mathbf{i}+b\mathbf{j})\cdot(3\mathbf{i}-4\mathbf{j}) &= 0\\ 3a-4b &= 0\\ b &= \frac{3}{4}a\\ a^2+b^2 &= 25^2\\ a^2+\left(\frac{3}{4}a\right)^2 &= 625\\ a^2+\frac{9}{16}a^2 &= 625\\ \frac{25}{16}a^2 &= 625\\ \frac{a^2}{16} &= 25\\ a^2 &= 400\\ a &= \pm 20\\ b &= \pm 15\end{aligned}$$
The two vectors are $(20\mathbf{i}+15\mathbf{j})$ and $(-20\mathbf{i}-15\mathbf{j})$.

22. First, find any vector perpendicular to $\begin{pmatrix}3\\2\end{pmatrix}$.
$$\begin{aligned}\begin{pmatrix}a\\b\end{pmatrix}\cdot\begin{pmatrix}3\\2\end{pmatrix} &= 0\\ 3a+2b &= 0\end{aligned}$$
There are infinitely many solutions to this, but for the vector equation of a line we really only concern ourselves with the direction that this represents so we may choose any solution that suits us, say $a=2$ and $b=-3$. This gives us the line
$$\mathbf{r}=\begin{pmatrix}4\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-3\end{pmatrix}$$
There are infinitely many equally correct answers, although few of them as simple and obvious as this. Note that Sadler's own answer is different.

23. $$\begin{aligned}(a\mathbf{i}+b\mathbf{j})\cdot(2\mathbf{i}+\mathbf{j}) &= 0\\ 2a+b &= 0\\ b &= -2a\\ a^2+b^2 &= 1^2\\ a^2+(-2a)^2 &= 1\\ a^2+4a^2 &= 1\\ 5a^2 &= 1\\ a^2 &= \frac{1}{5}\\ a &= \pm\frac{1}{\sqrt{5}}\\ b &= \mp\frac{2}{\sqrt{5}}\end{aligned}$$
The two vectors are $\left(\frac{1}{\sqrt{5}}\mathbf{i}-\frac{2}{\sqrt{5}}\mathbf{j}\right)$ and $\left(-\frac{1}{\sqrt{5}}\mathbf{i}+\frac{2}{\sqrt{5}}\mathbf{j}\right)$.
(We could have done this more simply: find a unit vector parallel to $\mathbf{i}-2\mathbf{j}$ or to $-\mathbf{i}+2\mathbf{j}$ ... almost a one-step solution.)

24. Use the approach outlined for questions 12 to 17 above:
$$\begin{aligned}\left|\begin{pmatrix}1\\-4\end{pmatrix}\right| &= \sqrt{1^2+(-4)^2}\\ &= \sqrt{17}\\ \left|\begin{pmatrix}2\\1\end{pmatrix}\right| &= \sqrt{2^2+1^2}\\ &= \sqrt{5}\\ \begin{pmatrix}1\\-4\end{pmatrix}\cdot\begin{pmatrix}2\\1\end{pmatrix} &= 1\times 2+-4\times 1\\ &= -2\\ \theta &= \cos^{-1}\frac{-2}{\sqrt{17}\sqrt{5}}\\ &= 103^\circ\end{aligned}$$
but this is obtuse so we need the supplementary angle: 77°.

25. $$\begin{aligned}\overrightarrow{AC} &= (7\mathbf{i}+2\mathbf{j})-(2\mathbf{i}+4\mathbf{j})\\ &= 5\mathbf{i}-2\mathbf{j}\\ \overrightarrow{BD} &= (4\mathbf{i}+1\mathbf{j})-(6\mathbf{i}+6\mathbf{j})\\ &= -2\mathbf{i}-5\mathbf{j}\\ \overrightarrow{AC}\cdot\overrightarrow{BD} &= 5\times -2+-2\times -5\\ &= 0\end{aligned}$$
$\therefore\ \overrightarrow{AC}$ is perpendicular to $\overrightarrow{BD}$.

26. (a) $$\begin{aligned}\overrightarrow{AC} &= (8\mathbf{i}+9\mathbf{j})-(4\mathbf{i}+7\mathbf{j})\\ &= 4\mathbf{i}+2\mathbf{j}\end{aligned}$$
(b) $$\begin{aligned}\overrightarrow{AB} &= (6\mathbf{i}+2\mathbf{j})-(4\mathbf{i}+7\mathbf{j})\\ &= 2\mathbf{i}-5\mathbf{j}\end{aligned}$$
(c) $$\begin{aligned}\overrightarrow{AC}\cdot\overrightarrow{AB} &= 4\times 2+2\times -5\\ &= -2\end{aligned}$$
(d) $$\begin{aligned}\angle\mathrm{CAB} &= \cos^{-1}\frac{\overrightarrow{AC}\cdot\overrightarrow{AB}}{|\overrightarrow{AC}||\overrightarrow{AB}|}\\ &= \cos^{-1}\frac{-2}{\sqrt{20}\sqrt{29}}\\ &= 95^\circ\end{aligned}$$

27. Start with the definition of dot product:
$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}||\mathbf{b}|\cos 45^\circ\\ \cos 45^\circ &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\\ \frac{1}{\sqrt{2}} &= \frac{3\times 4+-1\times y}{\sqrt{10}\sqrt{16+y^2}}\\ \frac{1}{\sqrt{2}} &= \frac{12-y}{\sqrt{10}\sqrt{16+y^2}}\\ 1 &= \frac{12-y}{\sqrt{5}\sqrt{16+y^2}}\\ 12-y &= \sqrt{5}\sqrt{16+y^2}\end{aligned}$$

square both sides, but note that our solution must satisfy $12 - y \geq 0$.

$$\begin{aligned}(12-y)^2 &= 5(16+y^2)\\ 144 - 24y + y^2 &= 80 + 5y^2\\ 4y^2 + 24y - 64 &= 0\\ y^2 + 6y - 16 &= 0\\ (y+8)(y-2) &= 0\\ y &= -8\\ \text{or } y &= 2\end{aligned}$$

Exercise 8D

1. $\mathbf{r}\cdot(3\mathbf{i}+4\mathbf{j}) = (2\mathbf{i}+3\mathbf{j})\cdot(3\mathbf{i}+4\mathbf{j})$
   $\mathbf{r}\cdot(3\mathbf{i}+4\mathbf{j}) = 6+12$
   $\mathbf{r}\cdot(3\mathbf{i}+4\mathbf{j}) = 18$

2. $\mathbf{r}\cdot(5\mathbf{i}-\mathbf{j}) = (-\mathbf{i}+7\mathbf{j})\cdot(5\mathbf{i}-\mathbf{j})$
   $\mathbf{r}\cdot(5\mathbf{i}-\mathbf{j}) = -5-7$
   $\mathbf{r}\cdot(5\mathbf{i}-\mathbf{j}) = -12$

3. $\mathbf{r}\cdot(2\mathbf{i}+3\mathbf{j}) = (-\mathbf{i}+2\mathbf{j})\cdot(2\mathbf{i}+3\mathbf{j})$
   $\mathbf{r}\cdot(2\mathbf{i}+3\mathbf{j}) = -2+6$
   $\mathbf{r}\cdot(2\mathbf{i}+3\mathbf{j}) = 4$

4. (A) $(6\mathbf{j})\cdot(\mathbf{i}+2\mathbf{j}) = 0+12 = 12$
   Point A lies on the line.
   (B) $(6\mathbf{i}+3\mathbf{j})\cdot(\mathbf{i}+2\mathbf{j}) = 6+6 = 12$
   Point B lies on the line.
   (C) $(10\mathbf{i})\cdot(\mathbf{i}+2\mathbf{j}) = 10+0 = 10$
   Point C does not lie on the line.
   (D) $(3\mathbf{i}+6\mathbf{j})\cdot(\mathbf{i}+2\mathbf{j}) = 3+12 = 15$
   Point D does not lie on the line.
   (E) $(-4\mathbf{i}+8\mathbf{j})\cdot(\mathbf{i}+2\mathbf{j}) = -4+16 = 12$
   Point E lies on the line.
   (F) $(14\mathbf{i}-\mathbf{j})\cdot(\mathbf{i}+2\mathbf{j}) = 14-2 = 12$
   Point F lies on the line.

5. (G) $(2\mathbf{i}+8\mathbf{j})\cdot(4\mathbf{i}-3\mathbf{j}) = 8-24 = -16$
   Point G lies on the line.
   (H) $(4\mathbf{i}-\mathbf{j})\cdot(4\mathbf{i}-3\mathbf{j}) = 16+3 = 19$
   Point H does not lie on the line.
   (I) $(8\mathbf{i}+16\mathbf{j})\cdot(4\mathbf{i}-3\mathbf{j}) = 32-48 = -16$
   Point I lies on the line.
   (J) $(-\mathbf{i}+4\mathbf{j})\cdot(4\mathbf{i}-3\mathbf{j}) = -4-12 = -16$
   Point J lies on the line.
   (K) $(-7\mathbf{i}-4\mathbf{j})\cdot(4\mathbf{i}-3\mathbf{j}) = -28+12 = -16$
   Point K lies on the line.
   (L) $(4\mathbf{i}+8\mathbf{j})\cdot(4\mathbf{i}-3\mathbf{j}) = 16-24 = -8$
   Point L does not lie on the line.

6. (U) $\begin{pmatrix}u\\2\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix} = 10$
   $2u+6 = 10$
   $u = 2$
   (V) $\begin{pmatrix}-10\\v\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix} = 10$
   $-20+3v = 10$
   $v = 10$
   (W) $\begin{pmatrix}w\\-4\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix} = 10$
   $2w-12 = 10$
   $w = 11$
   (X) $\begin{pmatrix}x\\-2\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix} = 10$
   $2x-6 = 10$
   $x = 8$

$$\text{(Y)}\quad \begin{pmatrix}5\\y\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix}=10$$
$$10+3y=10$$
$$y=0$$

$$\text{(Z)}\quad \begin{pmatrix}z\\6\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix}=10$$
$$2z+18=10$$
$$z=-4$$

7. (a)
$$\begin{aligned}\mathbf{r}\cdot(5\mathbf{i}+2\mathbf{j})&=(\mathbf{i}+\mathbf{j})\cdot(5\mathbf{i}+2\mathbf{j})\\ \mathbf{r}\cdot(5\mathbf{i}+2\mathbf{j})&=5+2\\ \mathbf{r}\cdot(5\mathbf{i}+2\mathbf{j})&=7\end{aligned}$$
(b)
$$\begin{aligned}\mathbf{r}\cdot(5\mathbf{i}+2\mathbf{j})&=7\\ (x\mathbf{i}+y\mathbf{j})\cdot(5\mathbf{i}+2\mathbf{j})&=7\\ 5x+2y&=7\end{aligned}$$

8. (a)
$$\begin{aligned}\mathbf{r}\cdot(2\mathbf{i}+5\mathbf{j})&=(2\mathbf{i}-\mathbf{j})\cdot(2\mathbf{i}+5\mathbf{j})\\ \mathbf{r}\cdot(2\mathbf{i}+5\mathbf{j})&=4-5\\ \mathbf{r}\cdot(2\mathbf{i}+5\mathbf{j})&=-1\end{aligned}$$
(b)
$$\begin{aligned}\mathbf{r}\cdot(2\mathbf{i}+5\mathbf{j})&=-1\\ (x\mathbf{i}+y\mathbf{j})\cdot(2\mathbf{i}+5\mathbf{j})&=-1\\ 2x+5y&=-1\end{aligned}$$

9. (a)
$$\begin{aligned}\mathbf{r}\cdot(\mathbf{i}-3\mathbf{j})&=(5\mathbf{i}+2\mathbf{j})\cdot(\mathbf{i}-3\mathbf{j})\\ \mathbf{r}\cdot(\mathbf{i}-3\mathbf{j})&=5-6\\ \mathbf{r}\cdot(\mathbf{i}-3\mathbf{j})&=-1\end{aligned}$$
(b)
$$\begin{aligned}\mathbf{r}\cdot(\mathbf{i}-3\mathbf{j})&=-1\\ (x\mathbf{i}+y\mathbf{j})\cdot(\mathbf{i}-3\mathbf{j})&=-1\\ x-3y&=-1\end{aligned}$$

10.
$$\begin{aligned}\mathbf{r}\cdot(\mathbf{i}-5\mathbf{j})&=(-2\mathbf{i}+\mathbf{j})\cdot(\mathbf{i}-5\mathbf{j})\\ \mathbf{r}\cdot(\mathbf{i}-5\mathbf{j})&=-2-5\\ \mathbf{r}\cdot(\mathbf{i}-5\mathbf{j})&=-7\\ (x\mathbf{i}+y\mathbf{j})\cdot(\mathbf{i}-5\mathbf{j})&=-7\\ x-5y&=-7\end{aligned}$$

11.
$$\begin{aligned}\mathbf{r}\cdot(2\mathbf{i}+\mathbf{j})&=(\mathbf{i}+2\mathbf{j})\cdot(2\mathbf{i}+\mathbf{j})\\ \mathbf{r}\cdot(2\mathbf{i}+\mathbf{j})&=2+2\\ \mathbf{r}\cdot(2\mathbf{i}+\mathbf{j})&=4\\ (x\mathbf{i}+y\mathbf{j})\cdot(2\mathbf{i}+\mathbf{j})&=4\\ 2x+y&=4\end{aligned}$$

12.
$$\mathbf{r}=2\mathbf{i}+3\mathbf{j}+\lambda(\mathbf{i}-4\mathbf{j})$$
is parallel to $(\mathbf{i}-4\mathbf{j})$ and
$$\mathbf{r}\cdot(8\mathbf{i}+2\mathbf{j})=5$$
is perpendicular to $(8\mathbf{i}+2\mathbf{j})$.
$(\mathbf{i}-4\mathbf{j})\cdot(8\mathbf{i}+2\mathbf{j})=8-8=0$ so $(\mathbf{i}-4\mathbf{j})$ is perpendicular to $(8\mathbf{i}+2\mathbf{j})$, hence
$$\mathbf{r}=2\mathbf{i}+3\mathbf{j}+\lambda(\mathbf{i}-4\mathbf{j})$$
is perpendicular to $(8\mathbf{i}+2\mathbf{j})$ and
$$\mathbf{r}=2\mathbf{i}+3\mathbf{j}+\lambda(\mathbf{i}-4\mathbf{j})$$
is parallel to
$$\mathbf{r}\cdot(8\mathbf{i}+2\mathbf{j})=5$$
□

13.
$$\begin{aligned}\mathbf{r}\cdot(8\mathbf{i}+5\mathbf{j})&=(-\mathbf{i}+3\mathbf{j})\cdot(8\mathbf{i}+5\mathbf{j})\\ \mathbf{r}\cdot(8\mathbf{i}+5\mathbf{j})&=-8+15\\ \mathbf{r}\cdot(8\mathbf{i}+5\mathbf{j})&=7\\ (x\mathbf{i}+y\mathbf{j})\cdot(8\mathbf{i}+5\mathbf{j})&=7\\ 8x+5y&=7\end{aligned}$$

14. L_2 is perpendicular to $(6\mathbf{i}-4\mathbf{j})$
$\therefore$ L_2 is perpendicular to $0.5(6\mathbf{i}-4\mathbf{j})=(3\mathbf{i}-2\mathbf{j})$
$\therefore$ L_2 is perpendicular to $\lambda(3\mathbf{i}-2\mathbf{j})$
$\therefore$ L_2 is perpendicular to $\mathbf{r}=5\mathbf{i}+2\mathbf{j}+\lambda(3\mathbf{i}-2\mathbf{j})$
$\therefore$ L_2 is perpendicular to L_1. □

15.
$$\begin{aligned}3x+4y&=7\\ (x\mathbf{i}+y\mathbf{j})\cdot(3\mathbf{i}+4\mathbf{j})&=7\end{aligned}$$
The line is perpendicular to $(3\mathbf{i}+4\mathbf{j})$. This vector has a magnitude of 5, so a unit vector with this direction is $(0.6\mathbf{i}+0.8\mathbf{j})$.

16.
$$\begin{aligned}-5x+12y&=11\\ (x\mathbf{i}+y\mathbf{j})\cdot(-5\mathbf{i}+12\mathbf{j})&=11\end{aligned}$$
The line is perpendicular to $(-5\mathbf{i}+12\mathbf{j})$. This vector has a magnitude of 13, so the unit vectors with this direction are $\pm\frac{1}{13}(-5\mathbf{i}+12\mathbf{j})$.

17.
$$\begin{aligned}3x-y&=5\\ (x\mathbf{i}+y\mathbf{j})\cdot(3\mathbf{i}-\mathbf{j})&=5\end{aligned}$$
The line is perpendicular to $(3\mathbf{i}-1\mathbf{j})$. This vector has a magnitude of $\sqrt{10}$, so the unit vectors with this direction are $\pm\frac{1}{\sqrt{10}}(3\mathbf{i}-1\mathbf{j})$.

18. L_1 is parallel to $\begin{pmatrix}1\\-2\end{pmatrix}$ so it is also parallel to $-\begin{pmatrix}1\\-2\end{pmatrix}=\begin{pmatrix}-1\\2\end{pmatrix}$.
L_2 is perpendicular to $\begin{pmatrix}-1\\2\end{pmatrix}$
$\therefore$ L_1 is perpendicular to L_2. □

$$\begin{aligned}\left(\begin{pmatrix}2\\8\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\end{pmatrix}\right)\cdot\begin{pmatrix}-1\\2\end{pmatrix}&=4\\ \begin{pmatrix}2+\lambda\\8-2\lambda\end{pmatrix}\cdot\begin{pmatrix}-1\\2\end{pmatrix}&=4\\ -1(2+\lambda)+2(8-2\lambda)&=4\\ -2-\lambda+16-4\lambda&=4\\ 14-5\lambda&=4\\ \lambda&=2\end{aligned}$$

$$\begin{aligned}\mathbf{r}&=\begin{pmatrix}2\\8\end{pmatrix}+2\begin{pmatrix}1\\-2\end{pmatrix}\\ &=\begin{pmatrix}2\\8\end{pmatrix}+\begin{pmatrix}2\\-4\end{pmatrix}\\ &=\begin{pmatrix}4\\4\end{pmatrix}\end{aligned}$$

Exercise 8E

1. Let A be the point of closest approach.
Let O be the position at 8am as illustrated.
Let t be the time of closest approach in hours after 8am.

$$\begin{aligned}\overrightarrow{PA} &= \overrightarrow{PO} + \overrightarrow{OA}\\ &= \overrightarrow{OA} - \overrightarrow{OP}\\ &= t(10\mathbf{i} + 5\mathbf{j}) - (25\mathbf{j} + 15\mathbf{j})\end{aligned}$$

Now because $\overrightarrow{PA}$ is perpendicular to $(10\mathbf{i} + 5\mathbf{j})$,

$$\begin{aligned}\overrightarrow{PA} \cdot (10\mathbf{i} + 5\mathbf{j}) &= 0\\ [t(10\mathbf{i} + 5\mathbf{j}) - (25\mathbf{j} + 15\mathbf{j})] \cdot (10\mathbf{i} + 5\mathbf{j}) &= 0\\ [t(10\mathbf{i} + 5\mathbf{j})] \cdot (10\mathbf{i} + 5\mathbf{j}) - (25\mathbf{j} + 15\mathbf{j}) \cdot (10\mathbf{i} + 5\mathbf{j}) &= 0\\ 125t - 325 &= 0\\ t &= \frac{13}{5}\end{aligned}$$

That is, the closest approach occurs 2.6 hours after 8am, or 10:36am. At that time

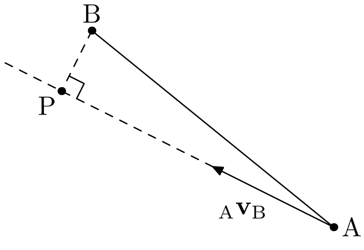
$$\begin{aligned}\overrightarrow{PA} &= \frac{13}{5}(10\mathbf{i} + 5\mathbf{j}) - (25\mathbf{j} + 15\mathbf{j})\\ &= \mathbf{i} - 2\mathbf{j}\\ PA &= \sqrt{1^2 + 2^2}\\ &= \sqrt{5}\end{aligned}$$

The closest approach is $\sqrt{5} \approx 2.24$km.

2. Apply a velocity of $-\mathbf{v}_B$ so as to treat B as stationary.

$$\begin{aligned}{}_A\mathbf{v}_B &= \mathbf{v}_A - \mathbf{v}_B\\ &= \begin{pmatrix}-10\\-2\end{pmatrix} - \begin{pmatrix}-2\\-6\end{pmatrix}\\ &= \begin{pmatrix}-8\\4\end{pmatrix}\end{aligned}$$

Let P be the point of closest approach of particle A towards B.



$$\begin{aligned}\overrightarrow{PB} &= \overrightarrow{PA} + \overrightarrow{AB}\\ &= \overrightarrow{AB} - \overrightarrow{AP}\\ &= \begin{pmatrix}-16\\13\end{pmatrix} - t\,{}_A\mathbf{v}_B\\ &= \begin{pmatrix}-16\\13\end{pmatrix} - t\begin{pmatrix}-8\\4\end{pmatrix}\end{aligned}$$

Now because $\overrightarrow{PB}$ is perpendicular to ${}_A\mathbf{v}_B$,

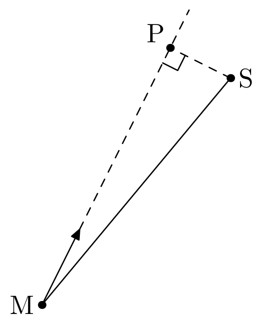
$$\begin{aligned}\overrightarrow{PB} \cdot {}_A\mathbf{v}_B &= 0\\ \left[\begin{pmatrix}-16\\13\end{pmatrix} - t\begin{pmatrix}-8\\4\end{pmatrix}\right] \cdot \begin{pmatrix}-8\\4\end{pmatrix} &= 0\\ \begin{pmatrix}-16\\13\end{pmatrix} \cdot \begin{pmatrix}-8\\4\end{pmatrix} - t\begin{pmatrix}-8\\4\end{pmatrix} \cdot \begin{pmatrix}-8\\4\end{pmatrix} &= 0\\ (-16 \times -8 + 13 \times 4) - t(-8 \times -8 + 4 \times 4) &= 0\\ 180 + 80t &= 0\\ t &= \frac{9}{4}\end{aligned}$$

The closest approach occurs at $t = 2.25$ seconds. At that time

$$\begin{aligned}\overrightarrow{PB} &= \begin{pmatrix}-16\\13\end{pmatrix} - \frac{9}{4}\begin{pmatrix}-8\\4\end{pmatrix}\\ &= \begin{pmatrix}2\\4\end{pmatrix}\\ PB &= \sqrt{2^2 + 4^2}\\ &= 2\sqrt{5}\end{aligned}$$

The closest approach is $2\sqrt{5} \approx 4.47$m.

3. Let M be the initial position of the mouse.
Let S be the initial position of the snake.
Let P be the position of closest approach.



$$\begin{aligned}\overrightarrow{PS} &= \overrightarrow{PM} + \overrightarrow{MS}\\ &= -t(\mathbf{i} + 2\mathbf{j}) + (5\mathbf{i} + 6\mathbf{j})\end{aligned}$$

$$\begin{aligned}\overrightarrow{PS} \cdot (\mathbf{i} + 2\mathbf{j}) &= 0\\ [-t(\mathbf{i} + 2\mathbf{j}) + (5\mathbf{i} + 6\mathbf{j})] \cdot (\mathbf{i} + 2\mathbf{j}) &= 0\\ -t(\mathbf{i} + 2\mathbf{j}) \cdot (\mathbf{i} + 2\mathbf{j}) + (5\mathbf{i} + 6\mathbf{j}) \cdot (\mathbf{i} + 2\mathbf{j}) &= 0\\ -5t + 17 &= 0\\ t &= \frac{17}{5}\end{aligned}$$

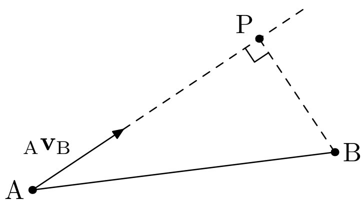
$$\begin{aligned}\overrightarrow{PS} &= -\frac{17}{5}(\mathbf{i} + 2\mathbf{j}) + (5\mathbf{i} + 6\mathbf{j})\\ &= (1.6\mathbf{i} - 0.8\mathbf{j})\\ PS &= \sqrt{1.6^2 + (-0.8)^2}\\ &= 1.8\text{m}\end{aligned}$$

The snake is more likely to catch the mouse than miss it.

4. Apply a velocity of $-\mathbf{v}_\mathrm{B}$ so as to treat B as stationary.

$$\begin{aligned}{}_\mathrm{A}\mathbf{v}_\mathrm{B} &= \mathbf{v}_\mathrm{A} - \mathbf{v}_\mathrm{B}\\ &= (3\mathbf{i}+4\mathbf{j}) - (-3\mathbf{i})\\ &= (6\mathbf{i}+4\mathbf{j})\mathrm{cm/s}\end{aligned}$$

Let P be the point of closest approach of particle A towards B.



$$\begin{aligned}\overrightarrow{\mathrm{PB}} &= \overrightarrow{\mathrm{PA}} + \overrightarrow{\mathrm{AB}}\\ &= -t_\mathrm{A}\mathbf{v}_\mathrm{B} + \overrightarrow{\mathrm{AB}}\\ &= -t(6\mathbf{i}+4\mathbf{j}) + (40\mathbf{i}+5\mathbf{j})\end{aligned}$$

Now because $\overrightarrow{\mathrm{PB}}$ is perpendicular to ${}_\mathrm{A}\mathbf{v}_\mathrm{B}$,

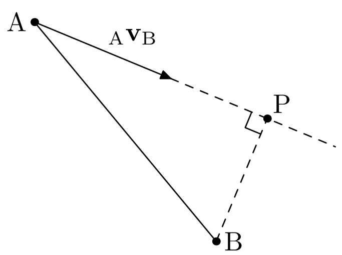
$$\begin{aligned}\overrightarrow{\mathrm{PB}}\cdot {}_\mathrm{A}\mathbf{v}_\mathrm{B} &= 0\\ [-t(6\mathbf{i}+4\mathbf{j}) + (40\mathbf{i}+5\mathbf{j})]\cdot(6\mathbf{i}+4\mathbf{j}) &= 0\\ -t(6\mathbf{i}+4\mathbf{j})\cdot(6\mathbf{i}+4\mathbf{j}) + (40\mathbf{i}+5\mathbf{j})\cdot(6\mathbf{i}+4\mathbf{j}) &= 0\\ -52t+260 &= 0\\ t &= 5\end{aligned}$$

The closest approach occurs at $t = 5$ seconds. At that time

$$\begin{aligned}\overrightarrow{\mathrm{PB}} &= -5(6\mathbf{i}+4\mathbf{j}) + (40\mathbf{i}+5\mathbf{j})\\ &= 10\mathbf{i} - 15\mathbf{j}\\ \mathrm{PB} &= 5\sqrt{13}\end{aligned}$$

The closest approach is $5\sqrt{13} \approx 18.0$cm.

5.
$$\begin{aligned}\overrightarrow{\mathrm{AB}} &= \mathbf{r}_\mathrm{B} - \mathbf{r}_\mathrm{A}\\ &= \begin{pmatrix}54\\-19\end{pmatrix} - \begin{pmatrix}30\\10\end{pmatrix}\\ &= \begin{pmatrix}24\\-29\end{pmatrix}\\ {}_\mathrm{A}\mathbf{v}_\mathrm{B} &= \mathbf{v}_\mathrm{A} - \mathbf{v}_\mathrm{B}\\ &= \begin{pmatrix}10\\-5\end{pmatrix} - \begin{pmatrix}-8\\7\end{pmatrix}\\ &= \begin{pmatrix}18\\-12\end{pmatrix}\end{aligned}$$



$$\begin{aligned}\overrightarrow{\mathrm{PB}} &= \overrightarrow{\mathrm{PA}} + \overrightarrow{\mathrm{AB}}\\ &= -t_\mathrm{A}\mathbf{v}_\mathrm{B} + \overrightarrow{\mathrm{AB}}\\ &= -t\begin{pmatrix}18\\-12\end{pmatrix} + \begin{pmatrix}24\\-29\end{pmatrix}\end{aligned}$$

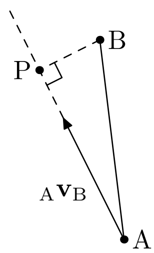
Now because $\overrightarrow{\mathrm{PB}}$ is perpendicular to ${}_\mathrm{A}\mathbf{v}_\mathrm{B}$,

$$\begin{aligned}\overrightarrow{\mathrm{PB}}\cdot {}_\mathrm{A}\mathbf{v}_\mathrm{B} &= 0\\ \left[-t\begin{pmatrix}18\\-12\end{pmatrix} + \begin{pmatrix}24\\-29\end{pmatrix}\right]\cdot\begin{pmatrix}18\\-12\end{pmatrix} &= 0\\ -t\begin{pmatrix}18\\-12\end{pmatrix}\cdot\begin{pmatrix}18\\-12\end{pmatrix} + \begin{pmatrix}24\\-29\end{pmatrix}\cdot\begin{pmatrix}18\\-12\end{pmatrix} &= 0\\ -468t+780 &= 0\\ t &= \frac{5}{3}\end{aligned}$$

The closest approach occurs at $t = \frac{5}{3}$ hours (i.e. at 4:40am). At that time

$$\begin{aligned}\overrightarrow{\mathrm{PB}} &= -\frac{5}{3}\begin{pmatrix}18\\-12\end{pmatrix} + \begin{pmatrix}24\\-29\end{pmatrix}\\ &= \begin{pmatrix}6\\-9\end{pmatrix}\\ \mathrm{PB} &= 3\sqrt{13}\mathrm{km}\end{aligned}$$

6.
$$\begin{aligned}\overrightarrow{\mathrm{AB}} &= \mathbf{r}_\mathrm{B} - \mathbf{r}_\mathrm{A}\\ &= (16\mathbf{i}+23\mathbf{j}) - (20\mathbf{i}-10\mathbf{j})\\ &= (-4\mathbf{i}+33\mathbf{j})\\ {}_\mathrm{A}\mathbf{v}_\mathrm{B} &= \mathbf{v}_\mathrm{A} - \mathbf{v}_\mathrm{B}\\ &= (4\mathbf{i}+5\mathbf{j}) - (6\mathbf{i}-3\mathbf{j})\\ &= (-2\mathbf{i}+8\mathbf{j})\end{aligned}$$



$$\begin{aligned}\overrightarrow{\mathrm{PB}} &= \overrightarrow{\mathrm{PA}} + \overrightarrow{\mathrm{AB}}\\ &= -t_\mathrm{A}\mathbf{v}_\mathrm{B} + \overrightarrow{\mathrm{AB}}\\ &= -t\,(-2\mathbf{i}+8\mathbf{j}) + (-4\mathbf{i}+33\mathbf{j})\end{aligned}$$

Because $\overrightarrow{\mathrm{PB}}$ is perpendicular to ${}_\mathrm{A}\mathbf{v}_\mathrm{B}$,

$$\begin{aligned}\overrightarrow{\mathrm{PB}}\cdot {}_\mathrm{A}\mathbf{v}_\mathrm{B} &= 0\\ [-t\,(-2\mathbf{i}+8\mathbf{j}) + (-4\mathbf{i}+33\mathbf{j})]\cdot(-2\mathbf{i}+8\mathbf{j}) &= 0\\ -t\,(-2\mathbf{i}+8\mathbf{j})\cdot(-2\mathbf{i}+8\mathbf{j}) + (-4\mathbf{i}+33\mathbf{j})\cdot\begin{pmatrix}-2\\8\end{pmatrix} &= 0\\ -68t+272 &= 0\\ t &= 4\end{aligned}$$

The closest approach occurs at $t = 4$ seconds. At that time

$$\begin{aligned}\overrightarrow{\text{PB}} &= -4\left(-2\mathbf{i} + 8\mathbf{j}\right) + \left(-4\mathbf{i} + 33\mathbf{j}\right)\\ &= \left(4\mathbf{i} + 1\mathbf{j}\right)\\ \text{PB} &= \sqrt{17}\text{m}\end{aligned}$$

7. Let P be the point where the perpendicular from A meets line L.

$$\begin{aligned}\overrightarrow{\text{OP}} &= -5\mathbf{i} + 22\mathbf{j} + \lambda_1\left(5\mathbf{i} - 2\mathbf{j}\right)\\ \overrightarrow{\text{AP}} &= \overrightarrow{\text{AO}} + \overrightarrow{\text{OP}}\\ &= -\left(14\mathbf{i} - 3\mathbf{j}\right) - 5\mathbf{i} + 22\mathbf{j} + \lambda_1\left(5\mathbf{i} - 2\mathbf{j}\right)\\ &= (5\lambda_1 - 19)\mathbf{i} + (-2\lambda_1 + 25)\mathbf{j}\end{aligned}$$

Because $\overrightarrow{\text{AP}}$ is perpendicular to L,

$$\begin{aligned}(5\mathbf{i} - 2\mathbf{j}) \cdot (5\lambda_1 - 19)\mathbf{i} + (-2\lambda_1 + 25)\mathbf{j} &= 0\\ 25\lambda_1 - 95 + 4\lambda_1 - 50 &= 0\\ 29\lambda_1 &= 145\\ \lambda_1 &= 5\end{aligned}$$

$$\begin{aligned}\therefore \qquad \overrightarrow{\text{AP}} &= (5 \times 5 - 19)\mathbf{i} + (-2 \times 5 + 25)\mathbf{j}\\ &= 6\mathbf{i} + 15\mathbf{j}\\ \text{AP} &= 3\sqrt{29}\end{aligned}$$

8. Let P be the point where the perpendicular from A meets line L.

$$\begin{aligned}\overrightarrow{\text{OP}} &= \begin{pmatrix}3\\-1\end{pmatrix} + \lambda_1\begin{pmatrix}3\\4\end{pmatrix}\\ \overrightarrow{\text{AP}} &= \overrightarrow{\text{AO}} + \overrightarrow{\text{OP}}\\ &= -\begin{pmatrix}11\\18\end{pmatrix} + \begin{pmatrix}3\\-1\end{pmatrix} + \lambda_1\begin{pmatrix}3\\4\end{pmatrix}\\ &= \begin{pmatrix}-8\\-19\end{pmatrix} + \lambda_1\begin{pmatrix}3\\4\end{pmatrix}\end{aligned}$$

Because $\overrightarrow{\text{AP}}$ is perpendicular to L,

$$\begin{aligned}\begin{pmatrix}3\\4\end{pmatrix} \cdot \left[\begin{pmatrix}-8\\-19\end{pmatrix} + \lambda_1\begin{pmatrix}3\\4\end{pmatrix}\right] &= 0\\ \begin{pmatrix}3\\4\end{pmatrix} \cdot \begin{pmatrix}-8\\-19\end{pmatrix} + \lambda_1\begin{pmatrix}3\\4\end{pmatrix} \cdot \begin{pmatrix}3\\4\end{pmatrix} &= 0\\ -100 + 25\lambda_1 &= 0\\ \lambda_1 &= 4\end{aligned}$$

$$\begin{aligned}\therefore \qquad \overrightarrow{\text{AP}} &= \begin{pmatrix}-8\\-19\end{pmatrix} + 4\begin{pmatrix}3\\4\end{pmatrix}\\ &= \begin{pmatrix}4\\-3\end{pmatrix}\\ \text{AP} &= 5\end{aligned}$$

9. Let P be the point where the perpendicular from A meets line L.

$$\begin{aligned}\overrightarrow{\text{OP}} &= \begin{pmatrix}-3\\0\end{pmatrix} + \lambda_1\begin{pmatrix}2\\-2\end{pmatrix}\\ \overrightarrow{\text{AP}} &= \overrightarrow{\text{AO}} + \overrightarrow{\text{OP}}\\ &= -\begin{pmatrix}-3\\8\end{pmatrix} + \begin{pmatrix}-3\\0\end{pmatrix} + \lambda_1\begin{pmatrix}2\\-2\end{pmatrix}\\ &= \begin{pmatrix}0\\-8\end{pmatrix} + \lambda_1\begin{pmatrix}2\\-2\end{pmatrix}\end{aligned}$$

Because $\overrightarrow{\text{AP}}$ is perpendicular to L,

$$\begin{aligned}\begin{pmatrix}2\\-2\end{pmatrix} \cdot \left[\begin{pmatrix}0\\-8\end{pmatrix} + \lambda_1\begin{pmatrix}2\\-2\end{pmatrix}\right] &= 0\\ \begin{pmatrix}2\\-2\end{pmatrix} \cdot \begin{pmatrix}0\\-8\end{pmatrix} + \lambda_1\begin{pmatrix}2\\-2\end{pmatrix} \cdot \begin{pmatrix}2\\-2\end{pmatrix} &= 0\\ 16 + 8\lambda_1 &= 0\\ \lambda_1 &= -2\end{aligned}$$

$$\begin{aligned}\therefore \qquad \overrightarrow{\text{AP}} &= \begin{pmatrix}0\\-8\end{pmatrix} - 2\begin{pmatrix}2\\-2\end{pmatrix}\\ &= \begin{pmatrix}-4\\-4\end{pmatrix}\\ \text{AP} &= 4\sqrt{2}\end{aligned}$$

10. (a) Substitute $\mathbf{r} = \begin{pmatrix}a\\b\end{pmatrix}$ into the equation of the line:

$$\begin{aligned}\begin{pmatrix}a\\b\end{pmatrix} \cdot \begin{pmatrix}-1\\2\end{pmatrix} &= 20\\ -a + 2b &= 20\\ a &= 2b - 20\end{aligned}$$

(b)
$$\begin{aligned}\overrightarrow{\text{AP}} &= \overrightarrow{\text{AO}} + \overrightarrow{\text{OP}}\\ &= -\begin{pmatrix}13\\4\end{pmatrix} + \begin{pmatrix}a\\b\end{pmatrix}\\ &= \begin{pmatrix}a - 13\\b - 4\end{pmatrix}\end{aligned}$$

(c) $\begin{pmatrix}-1\\2\end{pmatrix}$ is perpendicular to the line L, so if $\overrightarrow{\text{AP}}$ is to also be perpendicular to L then it must be parallel to $\begin{pmatrix}-1\\2\end{pmatrix}$.

(d) First substitute for a:

$$\begin{aligned}\overrightarrow{\text{AP}} &= \begin{pmatrix}a - 13\\b - 4\end{pmatrix}\\ &= \begin{pmatrix}2b - 33\\b - 4\end{pmatrix}\end{aligned}$$

Now because $\overrightarrow{\text{AP}}$ is parallel to $\begin{pmatrix}-1\\2\end{pmatrix}$

$$\begin{aligned}\begin{pmatrix}2b-33\\b-4\end{pmatrix} &= k\begin{pmatrix}-1\\2\end{pmatrix}\\ 2b-33 &= -k\\ \text{and}\qquad b-4 &= 2k\\ 4b-66 &= -2k\\ 5b-70 &= 0\\ b &= 14\\ a = 2b &= 20\\ &= 8\\ \overrightarrow{\text{AP}} &= \begin{pmatrix}a-13\\b-4\end{pmatrix}\\ &= \begin{pmatrix}8-13\\14-4\end{pmatrix}\\ &= \begin{pmatrix}-5\\10\end{pmatrix}\\ |\overrightarrow{\text{AP}}| &= 5\sqrt{5}\end{aligned}$$

11. Let P=$\begin{pmatrix}p\\q\end{pmatrix}$ be the point where the perpendicular from A=$\begin{pmatrix}x_1\\y_1\end{pmatrix}$ meets the line. Substitute $\mathbf{r} = \begin{pmatrix}p\\q\end{pmatrix}$ into the equation of the line:

$$\begin{aligned}\begin{pmatrix}p\\q\end{pmatrix} &= \begin{pmatrix}0\\c\end{pmatrix} + \lambda\begin{pmatrix}1\\m\end{pmatrix}\\ p &= \lambda\\ q &= c+\lambda m\\ &= c+mp\end{aligned}$$

$$\begin{aligned}\overrightarrow{\text{AP}} &= \overrightarrow{\text{AO}} + \overrightarrow{\text{OP}}\\ &= -\begin{pmatrix}x_1\\y_1\end{pmatrix} + \begin{pmatrix}p\\q\end{pmatrix}\\ &= \begin{pmatrix}p-x_1\\q-y_1\end{pmatrix}\end{aligned}$$

$\begin{pmatrix}1\\m\end{pmatrix}$ is parallel to the line, so a vector perpendicular to the line is $\begin{pmatrix}-m\\1\end{pmatrix}$. If $\overrightarrow{\text{AP}}$ is to also be perpendicular to the line then it must be parallel to $\begin{pmatrix}-m\\1\end{pmatrix}$. First substitute for q:

$$\begin{aligned}\overrightarrow{\text{AP}} &= \begin{pmatrix}p-x_1\\q-y_1\end{pmatrix}\\ &= \begin{pmatrix}p-x_1\\c+mp-y_1\end{pmatrix}\end{aligned}$$

Now because $\overrightarrow{\text{AP}}$ is parallel to $\begin{pmatrix}-m\\1\end{pmatrix}$

$$\begin{aligned}\begin{pmatrix}p-x_1\\c+mp-y_1\end{pmatrix} &= k\begin{pmatrix}-m\\1\end{pmatrix}\\ p-x_1 &= -mk\\ \text{and}\qquad c+mp-y_1 &= k\\ pm-mx_1 &= -m^2k\\ c-y_1+mx_1 &= k+m^2k\\ mx_1-y_1+c &= k(m^2+1)\\ k &= \frac{mx_1-y_1+c}{m^2+1}\\ \overrightarrow{\text{AP}} &= k\begin{pmatrix}-m\\1\end{pmatrix}\\ &= \frac{mx_1-y_1+c}{1+m^2}\begin{pmatrix}-m\\1\end{pmatrix}\\ |\overrightarrow{\text{AP}}| &= \left|\frac{mx_1-y_1+c}{1+m^2}\right|\sqrt{m^2+1}\\ &= \left|\frac{mx_1-y_1+c}{\sqrt{m^2+1}}\right|\end{aligned}$$

If we represent the line as $ax+by+d=0$, this translates to the vector equation

$$\mathbf{r}\cdot\begin{pmatrix}a\\b\end{pmatrix} = -d$$

where $\mathbf{r} = \begin{pmatrix}x\\y\end{pmatrix}$, so the line is perpendicular to $\begin{pmatrix}a\\b\end{pmatrix}$ and $\overrightarrow{\text{AP}}$ is parallel to $\begin{pmatrix}a\\b\end{pmatrix}$.

As before,

$$\overrightarrow{\text{AP}} = \begin{pmatrix}p-x_1\\q-y_1\end{pmatrix}$$

so

$$\begin{aligned}\begin{pmatrix}p-x_1\\q-y_1\end{pmatrix} &= k\begin{pmatrix}a\\b\end{pmatrix}\\ p-x_1 &= ak\\ \text{and}\qquad q-y_1 &= bk\\ ap-ax_1 &= a^2k\\ bq-by_1 &= b^2k\\ ap+bq-ax_1-by_1 &= k(a^2+b^2)\end{aligned}$$

but since P is a point on the line, $ap+bq+d=0$ or $ap+bq=-d$

$$\begin{aligned}-d-ax_1-by_1 &= k(a^2+b^2)\\ k &= -\frac{ax_1+by_1+d}{a^2+b^2}\\ \overrightarrow{\text{AP}} &= k\begin{pmatrix}a\\b\end{pmatrix}\\ &= -\frac{ax_1+by_1+d}{a^2+b^2}\begin{pmatrix}a\\b\end{pmatrix}\\ |\overrightarrow{\text{AP}}| &= \left|-\frac{ax_1+by_1+d}{a^2+b^2}\right|\sqrt{a^2+b^2}\\ &= \left|\frac{ax_1+by_1+d}{\sqrt{a^2+b^2}}\right|\end{aligned}$$

Miscellaneous Exercise 8

1. Cosine is positive in the 1st and 4th quadrants, so the smallest solution in the given domain is $x = 180 - 132 = 48$. The next is $x = 360 - 48 = 312$, then $x = 360 + 48 = 408$ and finally $x = 720 - 48 = 672$.

2. (a)
$$\begin{aligned} \mathbf{c}\cdot\mathbf{d} &= |\mathbf{c}||\mathbf{d}|\cos\theta \\ -5 &= 2\times 3\cos\theta \\ \cos\theta &= -\frac{5}{6} \\ \theta &= \cos^{-1} -\frac{5}{6} \\ &= 146° \end{aligned}$$

   (b) $\mathbf{c}\cdot\mathbf{c} = |\mathbf{c}|^2 = 2^2 = 4$

   (c) $\mathbf{d}\cdot\mathbf{d} = |\mathbf{d}|^2 = 3^2 = 9$

   (d)
$$\begin{aligned} (\mathbf{c}+\mathbf{d})\cdot(\mathbf{c}+\mathbf{d}) &= \mathbf{c}\cdot\mathbf{c} + 2\mathbf{c}\cdot\mathbf{d} + \mathbf{d}\cdot\mathbf{d} \\ &= 4 + 2\times(-5) + 9 \\ &= 3 \end{aligned}$$

   (e)
$$\begin{aligned} |\mathbf{c}+\mathbf{d}|^2 &= (\mathbf{c}+\mathbf{d})\cdot(\mathbf{c}+\mathbf{d}) \\ &= 3 \\ \therefore\quad |\mathbf{c}+\mathbf{d}| &= \sqrt{3} \end{aligned}$$

3. (a) True. $\mathbf{a}$ and $(\mathbf{b}-\mathbf{a})$ are perpendicular vectors so they must have a zero scalar product.

   (b) True. Proof:
$$\begin{aligned} \mathbf{a}\cdot(\mathbf{b}-\mathbf{a}) &= 0 && \text{(see part a)} \\ \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{a} &= 0 && \text{(expanding)} \\ \mathbf{a}\cdot\mathbf{b} &= \mathbf{a}\cdot\mathbf{a} \end{aligned}$$
□

   (c) Not necessarily true. $\mathbf{a}\cdot\mathbf{b} = k$ is the vector equation of a line perpendicular to $\mathbf{a}$ such that every point on the line can be substituted for $\mathbf{b}$ to satisfy the equation so we cannot conclude that $\mathbf{b} = \mathbf{a}$.

   (d) True. This follows from part (b).

4. Substitute $u = \frac{1}{2}\theta$. The L.H.S. becomes:
$$\begin{aligned} \frac{\sin 2u}{\cos u} &= \frac{2\sin u\cos u}{\cos u} \\ &= 2\sin u \\ &= 2\sin\left(\frac{1}{2}\theta\right) \\ &= \text{R.H.S.} \end{aligned}$$
□

5. (a) $\overrightarrow{\text{ED}} = \overrightarrow{\text{OD}} - \overrightarrow{\text{OE}} = -6\mathbf{i} + \mathbf{j}$

   (b) $\overrightarrow{\text{EF}} = \overrightarrow{\text{OF}} - \overrightarrow{\text{OE}} = 5\mathbf{i} + 5\mathbf{j}$

   (c) $\overrightarrow{\text{ED}}\cdot\overrightarrow{\text{EF}} = -6\times 5 + 1\times 5 = -25$

   (d)
$$\begin{aligned} \overrightarrow{\text{ED}}\cdot\overrightarrow{\text{EF}} &= |\overrightarrow{\text{ED}}||\overrightarrow{\text{EF}}|\cos\angle\text{DEF} \\ \cos\angle\text{DEF} &= \frac{\overrightarrow{\text{ED}}\cdot\overrightarrow{\text{EF}}}{|\overrightarrow{\text{ED}}||\overrightarrow{\text{EF}}|} \\ &= \frac{-25}{\sqrt{6^2+1^2}\sqrt{5^2+5^2}} \\ &= \frac{-25}{\sqrt{37}\sqrt{50}} \\ \angle\text{DEF} &= \cos^{-1}\frac{-25}{\sqrt{37}\sqrt{50}} \\ &= 126° \end{aligned}$$

6. (a) Already in the right form to read centre and radius directly. Centre$= (3, -2)$; radius$= 7$.

   (b) $|\mathbf{r} - (2\mathbf{i} + 7\mathbf{j})| = 11$.
   Centre$= (2, 7)$; radius$= 11$.

   (c) $(x-3)^2 + (y - -2)^2 = 4^2$
   Centre$= (3, -2)$; radius$= 4$.

   (d) $(x - -1)^2 + (y - -7)^2 = (2\sqrt{5})^2$
   Centre$= (-1, -7)$, radius$=2\sqrt{5}$.

   (e)
$$\begin{aligned} x^2 + y^2 - 8x &= 4y + 5 \\ x^2 - 8x + y^2 - 4y &= 5 \\ (x-4)^2 - 16 + (y-2)^2 - 4 &= 5 \\ (x-4)^2 + (y-2)^2 &= 25 \\ &= 5^2 \end{aligned}$$
   Centre$= (4, 2)$; radius$= 5$.

   (f)
$$\begin{aligned} x^2 + 6x + y^2 - 14y &= 42 \\ (x+3)^2 - 9 + (y-7)^2 - 49 &= 42 \\ (x - -3)^2 + (y-7)^2 &= 100 \\ &= 10^2 \end{aligned}$$
   Centre$= (-3, 7)$; radius$= 10$.

7. If solutions are not real then the discriminant $\Delta = b^2 - 4ac$ must be negative:
$$\begin{aligned} p^2 - 4\times 5\times 10 &< 0 \\ p^2 - 200 &< 0 \\ p^2 &< 200 \\ -10\sqrt{2} < p &< 10\sqrt{2} \end{aligned}$$

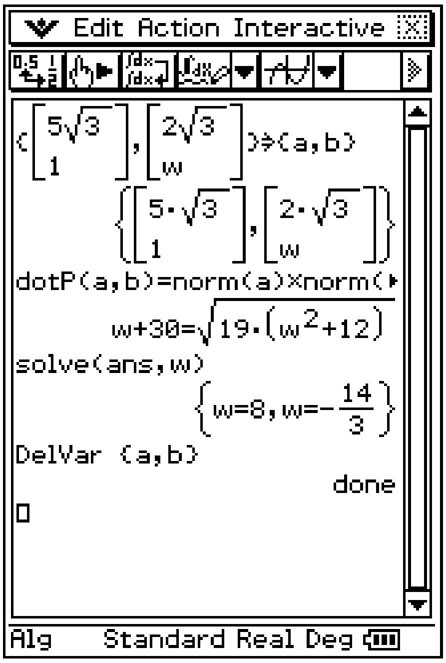
8. L_1 is parallel to $3\mathbf{i} - \mathbf{j}$ and L_2 is parallel to $2\mathbf{i} + 3\mathbf{j}$ so the angle between these vectors is the angle between the lines.
$$\begin{aligned} \cos\theta &= \frac{(3\mathbf{i}-\mathbf{j})\cdot(2\mathbf{i}+3\mathbf{j})}{|3\mathbf{i}-\mathbf{j}||2\mathbf{i}+3\mathbf{j}|} \\ &= \frac{3}{\sqrt{10}\sqrt{13}} \\ \theta &= \cos^{-1}\frac{3}{\sqrt{10}\sqrt{13}} \\ &= 75° \end{aligned}$$

9. Each of the pieces is internally continuous, so we need only concern ourselves with the endpoints

of the pieces.

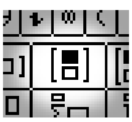
$$\begin{aligned}
f(-1) &= \lim_{x\to -1^+} f(x)\\
5(-1)+6 &= 2(-1)+a\\
a &= 3\\
\lim_{x\to 5^-} f(x) &= f(5)\\
2(5)+a &= b\\
b &= 13\\
f(5) &= \lim_{x\to 5^+} f(x)\\
13 &= 4(5)+c\\
c &= -7
\end{aligned}$$

10. This is very straightforward if you know how to use your calculator efficiently.



Going through this line by line:

- Use the curly brackets to make a list if you want to assign values to several variables at once. The curly brackets are on the `mth` tab or the `2D` tab. You could equally well have assigned the two variables individually; it makes little difference, but this way is a little more concise. (We could have done it without assigning variables at all by entering the vector values directly in the equation we want to solve, but this way is easier to follow.)
- To enter the values as column vectors, select the `CALC` page of the `2D` tab and choose the second icon:



- For the dot product of two vectors, use the `dotP` function from the `Action`→`Vector` menu and separate the two vectors with a comma.
- For the magnitude of a vector use the `norm` function under the `Action`→`Vector` menu. You can not use the absolute value `|a|` as this does not give the magnitude of a vector on the ClassPad.
- The third line in full reads `dotP(a,b)=norm(a)×norm(b)×cos(60)`
- I entered the equation I wanted to solve on one line and then solved it on the next line. This allows the whole thing to show on the calculator display which is useful for demonstration purposes, but we could equally well have done this all in one long line: `solve(dotP(a,b)=norm(a)×norm(b)×cos(60),w)`
- If your calculator is in `Cplx` mode you will get a warning that other solutions may exist. Because we are only interested in real solutions you may safely ignore this warning and it goes away if you switch to `Real` mode.
- The `DelVar` at the end deletes the variables we assigned. Although this is not strictly necessary it's not bad practice to delete variables when you've finished with them as it can reduce the occurance of problems if you later wish to re-use the variables used here without doing a `Clear All Variables` first.

You should also be able to do most (if not all) of this without a calculator:

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= |\mathbf{a}||\mathbf{b}|\cos\theta\\
\mathbf{a}\cdot\mathbf{b} &= (5\sqrt{3}\mathbf{i}+\mathbf{j})\cdot(2\sqrt{3}\mathbf{i}+w\mathbf{j})\\
&= 5\sqrt{3}\times 2\sqrt{3}+1\times w\\
&= 30+w
\end{aligned}$$

$$\begin{aligned}
|\mathbf{a}||\mathbf{b}|\cos\theta &= |5\sqrt{3}\mathbf{i}+\mathbf{j}||2\sqrt{3}\mathbf{i}+w\mathbf{j}|\cos 60^\circ\\
&= \sqrt{(5\sqrt{3})^2+1^2}\times\sqrt{(2\sqrt{3})^2+w^2}\times\frac{1}{2}\\
&= \sqrt{76}\sqrt{12+w^2}\frac{1}{\sqrt{4}}\\
&= \sqrt{\frac{76(12+w^2)}{4}}\\
&= \sqrt{228+19w^2}
\end{aligned}$$

$$\begin{aligned}
\therefore\quad 30+w &= \sqrt{228+19w^2}\\
(30+w)^2 &= 228+19w^2
\end{aligned}$$

(noting that $w+30 \geq 0$ so we discard any solution that gives $w < -30$)

$$\begin{aligned} 900 + 60w + w^2 &= 228 + 19w^2 \\ 18w^2 - 60w - 672 &= 0 \\ 3w^2 - 10w - 112 &= 0 \end{aligned}$$

Factors of $3 \times -112 = -336$ that add to -10 are -24 and 14

$$\begin{aligned} 3w^2 - 24w + 14w - 112 &= 0 \\ 3w(w-8) + 14(w-8) &= 0 \\ (3w+14)(w-8) &= 0 \\ w &= -\frac{14}{3} \\ \text{or} \quad w &= 8 \end{aligned}$$

11. $\log(100) = 2$ and $\ln(\mathrm{e}^{-3}) = -3$ so

$$\log(100) - \ln(\mathrm{e}^{-3}) = 2 - -3 = 5$$

12. (a)
$$\begin{aligned} \mathrm{e}^x + e^{x+1} &= 17 \\ \mathrm{e}^x + e^x \times e^1 &= 17 \\ \mathrm{e}^x(1+\mathrm{e}) &= 17 \\ \mathrm{e}^x &= \frac{17}{1+\mathrm{e}} \\ x &= \ln\frac{17}{1+\mathrm{e}} \\ &= \ln(17) - \ln(1+\mathrm{e}) \end{aligned}$$

(b)
$$\begin{aligned} \mathrm{e}^{2x+1} &= 50^{x-7} \\ 2x+1 &= \ln(50^{x-7}) \\ &= (x-7)\ln(50) \\ &= \ln(50)x - 7\ln(50) \\ 2x - \ln(50)x &= -7\ln(50) - 1 \\ x\,(2 - \ln(50)) &= -(7\ln(50)+1) \\ x &= \frac{-(7\ln(50)+1)}{2-\ln(50)} \\ &= \frac{7\ln(50)+1}{\ln(50)-2} \end{aligned}$$

13.
$$\begin{aligned} P &= 9\mathrm{e}^{(t+1)} \\ \frac{P}{9} &= \mathrm{e}^{(t+1)} \\ \ln\frac{P}{9} &= t+1 \\ t &= \ln\left(\frac{P}{9}\right) - 1 \end{aligned}$$

(a)
$$\begin{aligned} t &= \ln\left(\frac{180}{9}\right) - 1 \\ &= \ln(20) - 1 \\ &= 1.996 \end{aligned}$$

(b)
$$\begin{aligned} t &= \ln\left(\frac{3600}{9}\right) - 1 \\ &= \ln(400) - 1 \\ &= 4.991 \end{aligned}$$

(c)
$$\begin{aligned} t &= \ln\left(\frac{9e^3}{9}\right) - 1 \\ &= \ln(e^3) - 1 \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

14. (a) Product rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}x(2x+3) &= 1(2x+3) + 2(x) \\ &= 4x+3 \end{aligned}$$

(b) Quotient rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{2x+3} &= \frac{1(2x+3) - 2(x)}{(2x+3)^2} \\ &= \frac{3}{(2x+3)^2} \end{aligned}$$

(c) Product and chain rules:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}x(2x+3)^4 &= 1(2x+3)^4 + (4(2x+3)^3)(2)(x) \\ &= (2x+3)^4 + 8x(2x+3)^3 \\ &= (2x+3)^3(2x+3+8x) \\ &= (2x+3)^3(10x+3) \end{aligned}$$

15. Examining each piece for solutions:

- First piece:

$$\begin{aligned} -2x &= 1 \qquad x < 0 \\ x &= -\frac{1}{2} \end{aligned}$$

- Second piece:

$$\begin{aligned} x^2 &= 1 \qquad 0 \leq x < 2 \\ x &= \pm 1 \end{aligned}$$

  Discard the negative root as being outside the domain for this piece.

$$x = 1$$

- Third piece:

$$x = 2$$

- Fourth piece:

$$\begin{aligned} x+2 &= 1 \qquad x > 2 \\ x &= -1 \end{aligned}$$

  No solution in the domain for this piece.

The values of a that satisfy $\mathrm{f}(a) = 1$ are $a = -\frac{1}{2}, a = 1$ or $a = 2$.

16.
$$\begin{aligned}\tan 2x + \tan x &= 0\\ \tan 2x + \tan x &= 0\\ \frac{2\tan x}{1-\tan^2 x} + \tan x &= 0\\ \frac{2\tan x + \tan x(1-\tan^2 x)}{1-\tan^2 x} &= 0\\ 2\tan x + \tan x(1-\tan^2 x) &= 0 \quad [1-\tan^2 x \neq 0\\ \tan x(2+1-\tan^2 x) &= 0\\ \tan x(3-\tan^2 x) &= 0\end{aligned}$$

The first factor gives:

$$\begin{aligned}\tan x &= 0\\ x &= 0\\ \text{or}\quad x &= 180^\circ\\ \text{or}\quad x &= 360^\circ\end{aligned}$$

The second factor gives:

$$\begin{aligned}3-\tan^2 x &= 0\\ \tan x &= \pm\sqrt{3}\end{aligned}$$

giving solutions in all four quadrants 60° from the $x-$axis, i.e.

$$\begin{aligned}x &= 60^\circ\\ \text{or}\quad x &= 120^\circ\\ \text{or}\quad x &= 240^\circ\\ \text{or}\quad x &= 300^\circ\end{aligned}$$

17. Because **a** and **b** are perpendicular,

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= 0\\ (2\mathbf{i}-3\mathbf{j})\cdot(x\mathbf{i}+4\mathbf{j}) &= 0\\ 2x-12 &= 0\\ x &= 6\end{aligned}$$

Because **a** and **c** are parallel,

$$\begin{aligned}\mathbf{c} &= k\mathbf{a}\\ (9\mathbf{i}-y\mathbf{j}) &= k\,(2\mathbf{i}-3\mathbf{j})\\ 9 &= 2k\\ k &= 4.5\\ \text{and}\ -y &= k(-3)\\ y &= 3k\\ &= 13.5\end{aligned}$$

18. Let $z = a + b\mathrm{i}$ and $w = c + d\mathrm{i}$.

$$\begin{aligned}\mathrm{Re}(z) &= 6\\ \therefore\quad a &= 6\\ \mathrm{Re}(z)+\mathrm{Im}(w) &= -3\\ 6+d &= -3\\ d &= -9\\ 3z+6w &= zw\\ 3(6+b\mathrm{i})+6(c-9\mathrm{i}) &= (6+b\mathrm{i})(c-9\mathrm{i})\\ 18+3b\mathrm{i}+6c-54\mathrm{i} &= 6c-54\mathrm{i}+bc\mathrm{i}-9b\mathrm{i}^2\\ (18+6c)+(3b-54)\mathrm{i} &= (6c+9b)+(bc-54)\mathrm{i}\end{aligned}$$

equatinging the real components:

$$\begin{aligned}18+6c &= 6c+9b\\ 18 &= 9b\\ b &= 2\end{aligned}$$

equating the imaginary components:

$$\begin{aligned}3b-54 &= bc-54\\ 3b &= bc\\ c &= 3\\ \therefore\quad z &= 6+2\mathrm{i}\\ w &= 3-9\mathrm{i}\end{aligned}$$

19. L.H.S.:

$$\begin{aligned}&\sin\theta(\sin\theta+\sin 2\theta)\\ &= \sin\theta(\sin\theta+2\sin\theta\cos\theta)\\ &= \sin^2\theta(1+2\cos\theta)\\ &= (1-\cos^2\theta)(1+2\cos\theta)\\ &= 1+2\cos\theta-\cos^2\theta-2\cos^3\theta\\ &= \text{R.H.S.}\end{aligned}$$

□

20.
$$\begin{aligned}\mathbf{F}+\mathbf{P} &= (6\mathbf{i}+4\mathbf{j})+(2\mathbf{i}-7\mathbf{j})\\ &= (8\mathbf{i}-3\mathbf{j})\,\text{N}\\ |\mathbf{F}| &= \sqrt{6^2+4^2}\\ &= 2\sqrt{13}\\ &\approx 7.2\text{N}\\ |\mathbf{P}| &= \sqrt{2^2+(-7)^2}\\ &= \sqrt{53}\\ &\approx 7.3\text{N}\end{aligned}$$

The angle between the resultant and **F** is given by

$$\begin{aligned}(6\mathbf{i}+4\mathbf{j})\cdot(8\mathbf{i}-3\mathbf{j}) &= |\,(6\mathbf{i}+4\mathbf{j})\,||\,(8\mathbf{i}-3\mathbf{j})\,|\cos\theta\\ 48-12 &= 2\sqrt{13}\sqrt{8^2+(-3)^2}\cos\theta\\ 36 &= 2\sqrt{13}\sqrt{73}\cos\theta\\ \cos\theta &= \frac{36}{2\sqrt{13}\sqrt{73}}\\ \theta &= \cos^{-1}\frac{18}{\sqrt{13\times 73}}\\ &= 54^\circ\ \text{(nearest degree)}\end{aligned}$$

21. L_1 is parallel to $(10\mathbf{i}+4\mathbf{j})$
L_2 is parallel to $(-2\mathbf{i}+5\mathbf{j})$

$$\begin{aligned}(10\mathbf{i}+4\mathbf{j})\cdot(-2\mathbf{i}+5\mathbf{j}) &= 10\times(-2)+4\times 5\\ &= 0\\ \therefore\quad (10\mathbf{i}+4\mathbf{j}) &\perp (-2\mathbf{i}+5\mathbf{j})\\ \therefore\quad \mathrm{L}_1 &\perp \mathrm{L}_2\end{aligned}$$

□

22. (a)
$$\begin{aligned}R &= \sqrt{5^2+(-3)^2}\\ &= \sqrt{34}\end{aligned}$$
$$\begin{aligned}5\cos\theta - 3\sin\theta\\ &= \sqrt{34}\left(\frac{5}{\sqrt{34}}\cos\theta - \frac{3}{\sqrt{34}}\sin\theta\right)\\ &= \sqrt{34}\,(\cos\alpha\cos\theta - \sin\alpha\sin\theta)\\ &= \sqrt{34}\cos(\theta+\alpha)\end{aligned}$$
$$\begin{aligned}\text{where } \cos\alpha &= \frac{5}{\sqrt{34}}\\ \alpha &= 0.54\end{aligned}$$
$$\therefore \quad 5\cos\theta - 3\sin\theta = \sqrt{34}\cos(\theta+0.54)$$

(b) The minimum value is $-\sqrt{34}$ and this occurs when
$$\begin{aligned}\cos(\theta+0.54) &= -1\\ \theta+0.54 &= \pi\\ \theta &= \pi - 0.54\\ &= 2.60 \text{ (2 d.p.)}\end{aligned}$$

23. (a) Each piece of the function is individually continuous so it only remains to consider the continuity at $x=1$ and at $x=3$.

The function is discontinuous at $x=1$ because $\mathrm{f}(1)$ is not defined.

The function is continuous at $x=3$ because
$$\begin{aligned}\lim_{x\to3^-}\mathrm{f}(x) &= 2(3)-1\\ &= 5\\ \lim_{x\to3^+}\mathrm{f}(x) &= f(3)\\ &= 4(3)-7\\ &= 5\end{aligned}$$

$\mathrm{f}(x)$ is continuous $\forall x \in \Re, x \neq 1$.

(b) i. From the left,
$$\lim_{x\to0^-}\mathrm{f}'(x) = -1$$
but from the right
$$\lim_{x\to0^+}\mathrm{f}'(x) = 1$$
so $\mathrm{f}(x)$ is not differentiable at $x=0$.

ii. $\mathrm{f}(x)$ is differentiable at $x=0.5$ as it is continuous and the gradient of the curve for all $0<x<1$ is 1 so the derivative from the left is equal to the derivative from the right.

iii. $\mathrm{f}(x)$ is not continuous and therefore not differentiable at $x=1$.

iv. From the left,
$$\begin{aligned}\lim_{x\to3^-}\mathrm{f}'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(2x-1)\\ &= 2\end{aligned}$$
and from the right
$$\begin{aligned}\lim_{x\to3^+}\mathrm{f}'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(4x-7)\\ &= 4\end{aligned}$$
so $\mathrm{f}(x)$ is not differentiable at $x=3$.

24. Let P be the point on the line nearest A.
$$\begin{aligned}\overrightarrow{\mathrm{AP}} &= \overrightarrow{\mathrm{AO}} + \overrightarrow{\mathrm{OP}}\\ &= -\begin{pmatrix}-12\\-10\end{pmatrix} + \begin{pmatrix}-6\\12\end{pmatrix} + \lambda\begin{pmatrix}5\\1\end{pmatrix}\\ &= \begin{pmatrix}6\\22\end{pmatrix} + \lambda\begin{pmatrix}5\\1\end{pmatrix}\end{aligned}$$

$\overrightarrow{\mathrm{AP}}$ is perpendicular to the line so
$$\begin{aligned}\overrightarrow{\mathrm{AP}}\cdot\begin{pmatrix}5\\1\end{pmatrix} &= 0\\ \left(\begin{pmatrix}6\\22\end{pmatrix} + \lambda\begin{pmatrix}5\\1\end{pmatrix}\right)\cdot\begin{pmatrix}5\\1\end{pmatrix} &= 0\\ \begin{pmatrix}6\\22\end{pmatrix}\cdot\begin{pmatrix}5\\1\end{pmatrix} + \lambda\begin{pmatrix}5\\1\end{pmatrix}\cdot\begin{pmatrix}5\\1\end{pmatrix} &= 0\\ (30+22)+\lambda(25+1) &= 0\\ 52+26\lambda &= 0\\ \lambda &= -2\end{aligned}$$

$$\begin{aligned}\overrightarrow{\mathrm{AP}} &= \begin{pmatrix}6\\22\end{pmatrix} - 2\begin{pmatrix}5\\1\end{pmatrix}\\ \overrightarrow{\mathrm{AP}} &= \begin{pmatrix}-4\\20\end{pmatrix}\\ |\overrightarrow{\mathrm{AP}}| &= \sqrt{(-4)^2+20^2}\\ &= 4\sqrt{26}\end{aligned}$$

25. After 5 seconds
$$\begin{aligned}v &= \frac{200}{3}\left(1-\mathrm{e}^{-0.15\times5}\right)\\ &= 35.2\mathrm{ms}^{-1}\end{aligned}$$
Terminal velocity is given by
$$\begin{aligned}\lim t\to\infty\left(\frac{200}{3}\left(1-\mathrm{e}^{-0.15t}\right)\right) &= \frac{200}{3}(1-0)\\ &= \frac{200}{3}\mathrm{ms}^{-1}\\ &(\approx 66.7\mathrm{ms}^{-1})\end{aligned}$$

26.
$$\begin{aligned}x^2+2x+y^2-10y+a &= 0\\ (x+1)^2-1+(y-5)^2-25+a &= 0\\ (x+1)^2+(y-5)^2 &= 26-a\end{aligned}$$
If this is the equation of a circle then it must have a real, positive radius, so $26-a>0$ or $a<26$.

27. (a)
$$\begin{aligned}
|(-10\mathbf{i}+24\mathbf{j}+\lambda(5\mathbf{i}+\mathbf{j}))-(34\mathbf{i}+12\mathbf{j})| &\\
&= 2\sqrt{130}\\
|-44\mathbf{i}+12\mathbf{j}+\lambda(5\mathbf{i}+\mathbf{j})| &= 2\sqrt{130}\\
|(5\lambda-44)\mathbf{i}+(\lambda+12)\mathbf{j}| &= 2\sqrt{130}\\
(5\lambda-44)^2+(\lambda+12)^2 &= (2\sqrt{130})^2\\
25\lambda^2-440\lambda+1936 &\\
+\lambda^2+24\lambda+144 &= 520\\
26\lambda^2-416\lambda+1560 &= 0\\
\lambda^2-16\lambda+60 &= 0\\
(\lambda-6)(\lambda-10) &= 0
\end{aligned}$$

$$\begin{aligned}
&\lambda = 6\\
&\mathbf{r} = -10\mathbf{i}+24\mathbf{j}+6(5\mathbf{i}+\mathbf{j})\\
&\quad = 20\mathbf{i}+30\mathbf{j}\\
\text{or}\quad &\lambda = 10\\
&\mathbf{r} = -10\mathbf{i}+24\mathbf{j}+10(5\mathbf{i}+\mathbf{j})\\
&\quad = 40\mathbf{i}+34\mathbf{j}
\end{aligned}$$

The line intersects the circle at two points: $(20\mathbf{i}+30\mathbf{j})$ and $(40\mathbf{i}+34\mathbf{j})$.

(b)
$$\begin{aligned}
|(-\mathbf{i}+5\mathbf{j}+\lambda(3\mathbf{i}-\mathbf{j}))-(3\mathbf{i}+\mathbf{j})| &= \sqrt{5}\\
|-4\mathbf{i}+4\mathbf{j}+\lambda(3\mathbf{i}-\mathbf{j})| &= \sqrt{5}\\
|(3\lambda-4)\mathbf{i}+(4-\lambda)\mathbf{j}| &= \sqrt{5}\\
(3\lambda-4)^2+(4-\lambda)^2 &= 5\\
9\lambda^2-24\lambda+16+16-8\lambda+\lambda^2 &= 5\\
10\lambda^2-32\lambda+32 &= 5\\
10\lambda^2-32\lambda+27 &= 0
\end{aligned}$$

$$\begin{aligned}
\lambda &= \frac{32\pm\sqrt{(-32)^2-4\times10\times27}}{2\times10}\\
&= \frac{32\pm\sqrt{1024-1080}}{20}
\end{aligned}$$

Since the discriminant is negative it is clear that this has no real solutions. The line does not intersect the circle.

(c)
$$\begin{aligned}
|(-\mathbf{i}+7\mathbf{j}+\lambda(\mathbf{i}+3\mathbf{j}))-(4\mathbf{i}+2\mathbf{j})| &= 2\sqrt{10}\\
|-5\mathbf{i}+5\mathbf{j}+\lambda(\mathbf{i}+3\mathbf{j})| &= 2\sqrt{10}\\
|(\lambda-5)\mathbf{i}+(3\lambda+5)\mathbf{j}| &= 2\sqrt{10}\\
(\lambda-5)^2+(3\lambda+5)^2 &= (2\sqrt{10})^2\\
\lambda^2-10\lambda+25+9\lambda^2+30\lambda+25 &= 40\\
10\lambda^2+20\lambda+10 &= 0\\
\lambda^2+2\lambda+1 &= 0\\
(\lambda+1)^2 &= 0
\end{aligned}$$

$$\begin{aligned}
&\lambda = -1\\
&\mathbf{r} = -\mathbf{i}+7\mathbf{j}-(\mathbf{i}+3\mathbf{j})\\
&\quad = -2\mathbf{i}+4\mathbf{j}
\end{aligned}$$

The line intersects the circle at a single point: $(-2\mathbf{i}+4\mathbf{j})$ (that is, the line is a tangent to the circle).